Mathematics B
Master Solution Alternative Date Exam
Spring Semester 2019

Prof. Dr. Enrico De Giorgi[1]

5 February 2020

[1]Faculty of Mathematics and Statistics, University St. Gallen, Bodanstrasse 6, 9000 St. Gallen, Switzerland, email: enrico.degiorgi@unisg.ch.

Part I: Open Questions

Open questions require more than just applying a set of formulas or equations. Therefore, simply searching for the right formula is very likely not a successful strategy. By contrast, when reading the problem one should first identify key words that point to the most useful mathematical framework, and then, in a second stage, verify if the information provided allows to apply specific mathematical results. For example, if the text refers to "maximization" or "minimization", then very likely one has to apply mathematical optimisation and thus verify if this refers to constrained or to unconstrained optimisation. Translating the verbal description of a problem into an equivalent mathematical formulation of it is called *mathematical formalisation* and is a crucial step to make use of mathematics for solving problems. Open questions require that problems are formalised using mathematics before the adequate mathematical results and equations can be applied.

Exercise 1

(a) (9 points)

In this exercise the key word to understand which mathematical framework could be applied to solve the problem is "net profit...is maximal". This tells us that the problem is an optimisation problem and thus either an optimisation problem with constraints or without constraints. Constraints are conditions that limit the choice of the variables. However, in this case neither the independent variables p_1 and p_2, nor the dependent variables x and y (the demand in units for good 1 and good 2, respectively, which are functions of the prices), do face additional constraints. Therefore, we are facing an optimisation without constraints. The next question is which function needs to be maximised. The text mentions that the net profit should be maximal, so this is the objective to be maximised. We have:

$$\begin{aligned}
\text{profit} &= \text{revenue} - \text{costs} \\
P(p_1,p_2) &= \underbrace{(100-5\,p_1)\,p_1 + (200-4\,p_2)\,p_2}_{\text{revenue}} - \underbrace{K(100-5\,p_1, 200-4\,p_2)}_{\text{costs}} \\
&= (100-5\,p_1)\,p_1 + (200-4\,p_2)\,p_2 - \big((100-5\,p_1)^2 + (100-5\,p_1)\,(200-4\,p_2) + (200-4\,p_2)^2\big) \\
&= -30\,p_1^2 - 20\,p_1\,p_2 - 20\,p_2^2 + 2100\,p_1 + 2200\,p_2 - 70000.
\end{aligned}$$

A local maximum $(p_1^\star, p_2^\star)$ of function P must satisfy the necessary conditions:

$$P_{p_1}(p_1^\star, p_2^\star) = 0 \text{ and } P_{p_2}(p_1^\star, p_2^\star) = 0.$$

It follows that:

$$\left\{ \begin{array}{l} -60\,p_1^\star - 20\,p_2^\star + 2100 = 0 \\ -20\,p_1^\star - 40\,p_2^\star + 2200 = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{ll} 6\,p_1^\star + 2\,p_2^\star = 210 & (1) \\ 2\,p_1^\star + 4\,p_2^\star = 220 & (2) \end{array} \right. .$$

Two times (1) minus (2) implies:

$$10\,p_1^\star = 200 \Leftrightarrow p_1^\star = 20.$$

Plugging this result into (2) implies:

$$p_2^\star = 45.$$

It follows that there is a unique stationary point $(p_1^\star, p_2^\star) = (20, 45)$ of P, and thus a unique candidate for a local maximum.

Next we need to check the sufficient conditions for a local maximum. We have:

$$\left\{\begin{array}{c} P_{p_1,p_1}(p_1^\star,p_2^\star)<0 \\ P_{p_2,p_2}(p_1^\star,p_2^\star)<0 \\ P_{p_1,p_1}(p_1^\star,p_2^\star)\,P_{p_2,p_2}(p_1^\star,p_2^\star)-(P_{p_1,p_2}(p_1^\star,p_2^\star))^2>0 \end{array}\right. \quad \Rightarrow (p_1^\star,p_2^\star) \text{ is a maximum.}$$

We have:

$$\begin{aligned} P_{p_1,p_1}(p_1^\star,p_2^\star) &= -60<0, \\ P_{p_2,p_2}(p_1^\star,p_2^\star) &= -40<0, \\ P_{p_1,p_2}(p_1^\star,p_2^\star) &= -20<0. \end{aligned}$$

Therefore,

$$P_{p_1,p_1}(p_1^\star,p_2^\star)\,P_{p_2,p_2}(p_1^\star,p_2^\star)-(P_{p_1,p_2}(p_1^\star,p_2^\star))^2=(-60)\,(-40)-(-20)^2>0$$

and thus

$$(p_1^\star,p_2^\star)=(20,45)$$

is a local maximum of P.

Therefore, the maximal net profit is

$$P(20,45)=500.$$

(b) (9 points)

In this exercise the key word to understand which mathematical framework could be applied to solve the problem is "costs...are minimal". This tells us that the problem is an optimisation problem and thus either an optimisation problem with constraints or without constraints. Constraints are conditions that limit the choice of the variables. Indeed, in this case the amounts of wool x and y are constrained by the condition that the production of shirts should require exactly 100 tons of wool. Therefore, we are facing an optimisation with constraints. The next question is which function needs to be minimised and how the constraint looks like. The text mentions that the total costs for the wool should be minimal, so $K(x,y)=K_1(x)+K_2(y)$ is the objective function to be minimised. The constraint corresponds to the total amount of wool required, consisting of the amounts x and y by supplier 1 and 2, respectively, and is given by the equation $x+y=100$. It follows that we need to solve the following problem:

$$\min K(x,y)=K_1(x)+K_2(y)=\frac{3}{4}\,x^2+400+\frac{11}{4}\,y^2+10\,y+300=\frac{3}{4}\,x^2+\frac{11}{4}\,y^2+10\,y+700$$

subject to the constraint:

$$x+y=100.$$

To solve this problem, we apply the substitution method, as the form of the constraint easily allows us to explicitly write x as function of y, i.e.,

$$x=100-y.$$

We plug this result into K and obtain:

$$k(y)=K(100-y,y)=\frac{3}{4}\,(100-y)^2+\frac{11}{4}\,y^2+10\,y+700=\frac{7}{2}\,y^2-140\,y+8200.$$

Function k is a quadratic function with a strictly positive coefficient in front of the quadratic terms.

Therefore, k is strictly convex and any stationary point $y^\star$ is also a global minimum. Therefore, we solve:

$$k'(y^\star) = 0 \Leftrightarrow 7\,y^\star - 140 = 0 \Leftrightarrow y^\star = 20.$$

It follows that $y^\star = 20$ is a global minimum of k, and thus

$$(x^\star, y^\star) = (100 - y^\star, y^\star) = (80, 20)$$

is global minimum of K given the constraint $x + y = 100$. Moreover,

$$K(80, 20) = 6800$$

is the minimal production cost.

(c) (8 points)

In this exercise, we are confronted with constraints in relation to the total running times of machines. Indeed, if we define the total number units of product P_i produced in a day by x_i, then

$$10\,x_1 + 7\,x_2 + 8\,x_3 + 4\,x_4$$

is the total number of minutes needed by machine M_1 to produce those units. Because each machine runs for 16 hours, i.e., $16 \cdot 60 = 960$ minutes, then for machine M_1 we face the following constraint:

$$10\,x_1 + 7\,x_2 + 8\,x_3 + 4\,x_4 = 960.$$

Similarly, for machines M_2, M_3 and M_4 we have:

$$5\,x_1 + 5\,x_2 + 4\,x_3 + 11\,x_4 = 960,$$

$$8\,x_1 + 9\,x_2 + 6\,x_3 + 6\,x_4 = 960,$$

$$10\,x_1 + m\,x_2 + 9\,x_3 + 9\,x_4 = 960,$$

respectively.

The four equations lead to a system of linear equations:

$$\left\{\begin{array}{rcl} 10\,x_1 + 7\,x_2 + 8\,x_3 + 4\,x_4 & = & 960 \\ 5\,x_1 + 5\,x_2 + 4\,x_3 + 11\,x_4 & = & 960 \\ 8\,x_1 + 9\,x_2 + 6\,x_3 + 6\,x_4 & = & 960 \\ 10\,x_1 + m\,x_2 + 9\,x_3 + 9\,x_4 & = & 960 \end{array}\right. ,$$

which we solve using Gaussian elimination. We have:

$$(A, \mathbf{b}) \;=\; \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} 10 & 7 & 8 & 4 & 960 \\ 5 & 5 & 4 & 11 & 960 \\ 8 & 9 & 6 & 6 & 960 \\ 10 & m & 9 & 4 & 960 \end{array}\right) \end{array} \begin{array}{l} :(10) \\ \\ \\ \\ \end{array}$$

$$\rightarrow \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0.7 & 0.8 & 0.4 & 96 \\ 5 & 5 & 4 & 11 & 960 \\ 8 & 9 & 6 & 6 & 960 \\ 10 & m & 9 & 4 & 960 \end{pmatrix} \begin{matrix} \\ \\ -5\,(I) \\ -8\,(I) \\ -10\,(I) \end{matrix}$$

$$\rightarrow \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0.7 & 0.8 & 0.4 & 96 \\ 0 & 1.5 & 0 & 9 & 480 \\ 0 & 3.4 & -0.4 & 2.8 & 192 \\ 0 & m-7 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ :(1.5) \\ \\ \\ \end{matrix}$$

$$\rightarrow \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0.7 & 0.8 & 0.4 & 96 \\ 0 & 1 & 0 & 6 & 320 \\ 0 & 3.4 & -0.4 & 2.8 & 192 \\ 0 & m-7 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} \\ -0.7\,(II) \\ \\ -3.4\,(II) \\ -(m-7)\,(II) \end{matrix}$$

$$\rightarrow \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0.8 & -3.8 & -128 \\ 0 & 1 & 0 & 6 & 320 \\ 0 & 0 & -0.4 & -17.6 & -896 \\ 0 & 0 & 1 & 42-6\,m & -320\,m+2240 \end{pmatrix} \begin{matrix} \\ \\ \\ :(-0.4) \\ \\ \end{matrix}$$

$$\rightarrow \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0.8 & -3.8 & -128 \\ 0 & 1 & 0 & 6 & 320 \\ 0 & 0 & 1 & 44 & 2240 \\ 0 & 0 & 1 & 42-6\,m & -320\,m+2240 \end{pmatrix} \begin{matrix} \\ -0.8\,(III) \\ -6\,(III) \\ \\ -(III) \end{matrix}$$

$$\rightarrow \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & -39 & -1920 \\ 0 & 1 & 0 & 6 & 320 \\ 0 & 0 & 1 & 44 & 2240 \\ 0 & 0 & 1 & -6\,m-2 & -320\,m \end{pmatrix}$$

If $-6\,m-2=0$, i.e., $m=-\frac{1}{3}$, then $\text{rg}(A)=3<\text{rg}(A,\mathbf{b})=4$, since $-320\cdot\left(-\frac{1}{3}\right)\neq 0$. In this case, the system $A\,\mathbf{x}=\mathbf{b}$ has no solution.

Otherwise, for all $m\neq -\frac{1}{3}$, and thus also for all $m>0$, the system has exactly one solution $\mathbf{x}=(x_1,x_2,x_3,x_4)^T$. Be aware however that this solution not necessarily is composed of integers (as one would expect, given that x_i is the number of units of product P_i).

(d) (6 + 4 points)
The key aspect in this exercise is to understand the evolution of variable V_n, i.e., the amount available on the bank account at the end of the n-th year.

The goal is to clearly specify all transactions during the $(n+1)$-th year. First of all, at the beginning of the $(n+1)$-th year the amount is increased by CHF 5000. Then, Dora withdraws 60% by the end of June. Additionally, interests are paid over the year. There are two possible solutions to the exercise, based on how to understand the interest payments over the course of the year.

Possibility 1: Money weighted interests

The evoluation of wealth is decomposed into the evoluation without interest plus the interest for the first half of the year, plus those for the second half of the year.

(d1) (6 points)

$$\begin{aligned}
V_{n+1} &= \underbrace{(V_n + 5'000) \cdot 0.4}_{\text{evolution of wealth without interest}} \\
&\quad + \underbrace{\frac{1}{2} \cdot 0.0125 \cdot (V_n + 5'000)}_{\text{interest for the first semester}} \\
&\quad + \underbrace{\frac{1}{2} \cdot 0.0125 \cdot 0.4 \cdot (V_n + 5'000)}_{\text{interest for the second semester}} \\
&= 0.40875\, V_n + 2'043.75
\end{aligned}$$

Furthermore, $V_0 = 0$. We obtain a linear difference equation of first order with $A = 0.40875$ and $B = 2'043.75$. To find the explicit representation for the evoluation of wealth, we define

$$V^* = \frac{2'043.75}{1 - 0.40875} \approx 3'456.65$$

It follows

$$\begin{aligned}
V_n &= 0.40875^n\,(-V^*) + V^* \\
&= 3'456.65 \cdot (1 - 0.40875^n)
\end{aligned}$$

(d2) (4 points)

From the first part of the exercise, we obain long term savings (i.e. $\lim_{n\to\infty} V_n$) of $3'456.65$. To answer the question, when the savings exceed $3'400$ for the first time at the end of the year, we solve the inequality

$$\begin{aligned}
& & V_n &\geq 3400 \\
&\Leftrightarrow & V^*(1 - 0.40875^n) &\geq 3400 \\
&\Leftrightarrow & 1 - 0.40875^n &\geq \frac{3400}{V^*} \\
&\Leftrightarrow & n &\geq \frac{\ln(1 - \frac{3400}{V^*})}{\ln(0.40875)} \approx 4.595.
\end{aligned}$$

It follows that it takes at least 5 years.

Possibility 2: Time weighted interest payments

(d1) (6 points)

The idea is to split the evoluation of wealth into two time periods. First, we consider the evoluation of wealth until the end of June each year. That means, we get interests for half a year:

$$(V_n + 5000) + (V_n + 5000)\,\frac{1}{2} \cdot 1.25\% = (V_n + 5000) \cdot \left(1 + \frac{1}{2} \cdot 1.25\%\right).$$

Because 60% of this amount is withdrawn at the beginning of July, only 40% of the amount above remains on the account. Therefore, at the end of the $(n+1)$-th year, the amount on the account is:

$$V_{n+1} = 40\% \cdot (V_n + 5000) \cdot \left(1 + \frac{1}{2} \cdot 1.25\%\right) \cdot \left(1 + \frac{1}{2} \cdot 1.25\%\right) = 0.4\,(V_n + 5000)\,\left(1 + \frac{1}{2} \cdot 1.25\%\right)^2.$$

Using that

$$1 + \frac{1}{2} \cdot 1.25\% = 1 + \frac{1}{2} \cdot \frac{5}{400} = 1 + \frac{5}{800} = 1 + \frac{1}{160} = \frac{161}{160},$$

it follows:

$$V_{n+1} = \underbrace{0.4 \left(\frac{161}{160}\right)^2}_{=A} V_n + \underbrace{0.5 \cdot 5000 \left(\frac{161}{160}\right)^2}_{=B} \approx 0.4050156 \cdot V_n + 2025.078.$$

It follows that sequence $\{V_n\}_{n\geq 0}$ is described by a linear first order difference equation with $V_0 = 0$. Therefore,

$$V_n = A^n\,(V_0 - V^\star) + V^\star = (1 - A^n)\,V^\star$$

where

$$V^\star = \frac{B}{1-A} = \frac{2025.078}{1 - 0.4050156} \approx 3403.582.$$

Therefore,

$$V_n = 3403.582 \cdot (1 - 0.4050156^n).$$

(d2) (4 points)

First of all,

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} 3403.582 \cdot (1 - 0.4050156^n) = 3403.582.$$

Moreover,

$$\begin{aligned} V_n \geq 3400 \quad &\Leftrightarrow \quad 3403.582 \cdot (1 - 0.4050156^n) \geq 3400 \\ &\Leftrightarrow \quad 1 - 0.4050156^n \geq \frac{3400}{3403.582} \\ &\Leftrightarrow \quad 0.4050156^n \leq 1 - \frac{3400}{3403.582} \\ &\Leftrightarrow \quad n \geq \frac{\ln\left(1 - \frac{3400}{3403.582}\right)}{\ln(0.4050156)} \approx 7.586233. \end{aligned}$$

It follows that the amount on the bank account will reach CHF 3400 after 8 years.

In case the coefficients of the hint have been used for part (d2), it follows that the long term wealth stabilizes *at* $3'400$*, and that it takes at least 7 years to reach to amount of* $3'400$.

Part II: Multiple-choice Questions

Exercise 2

	(a)	(b)	(c)	(d)
Question 1	☒	☐	☐	☐
Question 2	☐	☐	☐	☒
Question 3	☒	☐	☐	☐
Question 4	☐	☐	☐	☒
Question 5	☐	☒	☐	☐
Question 6	☐	☒	☐	☐
Question 7	☒	☐	☐	☐
Question 8	☐	☐	☐	☒
Question 9	☐	☐	☒	☐
Question 10	☐	☒	☐	☐

Q1. Answer is (a). Because (x_0, y_0) is a stationary point, then it fulfils $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. The point (x_0, y_0) is a local extreme point when

$$f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0$$

holds. Because $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$ this condition is equivalent to the condition in (a).

Q2. Answer is (d). Because $x + y = 0$, then $y = -x$ and $f(x, y) = f(x, -x) = x^2 \cdot x = x^3$. Therefore, given the constraint $x + y = 0$, the objective is the cubic function $x \mapsto x^3$, which has no extreme points. Therefore, answer (d) is correct.

Q3. (a). P_1 is an extreme point under the given constraint because the tangent lines to the contour line $f(x, y) = 4 + \ln(2)$ and to the curve $\varphi(x, y) = 0$, respectively, are identical. Moreover,P_1 is a maximum, because moving away from P_1 along the curve $\varphi(x, y) = 0$ always leads to lower values of f.

Q4. (d). We have:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, x = \frac{-1}{\mu} \int_b^a \mu\, f(x)\, dx = -\frac{k_1}{\mu}$$

and

$$\int_a^b g(x)\, dx = \frac{1}{\rho} \int_a^b \rho\, g(x)\, dx = \frac{k_2}{\rho}.$$

It follows that:

$$\int_a^b \left(g(x) + f(x)\right)\, dx = \int_a^b g(x)\, dx + \int_a^b f(x)\, dx = \frac{k_2}{\rho} - \frac{k_1}{\mu}.$$

Q5. (b). We have:

$$f(x) = \int_x^4 (3\,t^2 - t + 1)\, dt = -\int_4^x (3\,t^2 - t + 1)\, dt.$$

It follows that:

$$f(0) = \int_0^4 (3\,t^2 - t + 1)\, dt = \left[t^3 - \frac{1}{2}\,t^2 + t\right]_0^4 = 60.$$

Moreover,

$$f'(x) = -(3\,x^2 - x + 1),$$
$$f''(x) = -6\,x + 1,$$
$$f'''(x) = -6.$$

Therefore, $f'(0) = -1$, $f''(0) = 1$, and $f'''(0) = -6$.

Q6. (b). We have:

$$(A+B)^2 = (A+B)\,(A+B) = A\,A + A\,B + B\,A + B\,B = A^2 + A\,B + B\,A + B^2 = A\,B + B^2 + A^2 + B\,A.$$

Q7. (a). If M is idempotent, then

$$M^2 = M.$$

Because M is also invertible, we multiply the latter equation by M^{-1} and obtain:

$$M^{-1}\,M^2 = \underbrace{M^{-1}\,M}_{=I} \Leftrightarrow \underbrace{M^{-1}\,M}_{=I}\,M = I \Leftrightarrow I\,M = I \Leftrightarrow M = I.$$

Therefore, the identity matrix I is the only invertible matrix that is also idempotent. It follows that answers (b)-(d) are correct and (a) is false.

Q8. (d). The gradient of f at a given point is perpendicular to the contour line of f at that point. Only vector $\mathbf{x}_4$ satisfies this property.

Q9. (c). Answer (a), (b) and (d) are true. By contrast, (c) is generally wrong, because A could be a linear dependent system even if $\mathbf{a}_k$ is linearly independent from $\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}$.

Q10. (b). Because $\mathrm{rg}(A) = \mathrm{rg}(A, \mathbf{b}) = 4$, then $A\,\mathbf{x} = \mathbf{b}$ possesses either one or infinitely many solutions. Because $\mathrm{rg}(A) = 4 < 7 = n$, then the system possesses infinitely many solutions and the solution space has dimension $n - \mathrm{rg}(A) = 7 - 4 = 3$.

Exercise 3

	(a)	(b)	(c)	(d)
Question 1	☒	☐	☐	☐
Question 2	☐	☒	☐	☐
Question 3	☐	☒	☐	☐
Question 4	☐	☐	☒	☐
Question 5	☐	☐	☒	☐
Question 6	☐	☒	☐	☐
Question 7	☐	☐	☐	☒
Question 8	☒	☐	☐	☐

Q1. (a). We solve the integral using the substitution rule. We set $y = \cos(n\,x)$, then $\frac{dy}{dx} = -n\,\sin(n\,x)$. It follows:

$$\int \sin(n\,x)\,e^{\cos(n\,x)}\,dx = \int -\frac{1}{n}\,e^y\,dy = -\frac{1}{n}\,e^y + C = -\frac{1}{n}\,e^{\cos(n\,x)} + C.$$

We have:

$$\int_0^{\frac{\pi}{2\,n}} \sin(n\,x)\,e^{\cos(n\,x)}\,dx = -\frac{1}{n}\,\left[e^{\cos(n\,x)}\right]_0^{\frac{\pi}{2\,n}} = -\frac{1}{n} + \frac{1}{n}\,e = \frac{1}{n}\,(e-1).$$

Q2. (b). f is a density function if and only if $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$. First of all, $f(x) \geq 0$ for all x if and only if $c \geq 1$. Moreover,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_c^{\infty} \frac{1}{x^2}\,\ln(x)\,dx = \lim_{N\to\infty}\left[-\frac{\ln(x)}{x} - \frac{1}{x}\right]_c^N = \frac{\ln(c)}{c} + \frac{1}{c} = 1 \Leftrightarrow c = 1.$$

Q3. (b). The direction of the steepest increase of f in (x_0, y_0) is given by the gradient of f in (x_0, y_0). Therefore,

$$\text{grad}f(x_0, y_0) = \begin{pmatrix} x_0^{-0.75}\,y_0^{0.75} \\ 3\,x_0^{0.25}\,y_0^{-0.25} \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 48 \end{pmatrix}$$

for some $\lambda > 0$. It follows that:

$$\frac{3\,x_0^{0.25}\,y_0^{-0.25}}{x_0^{-0.75}\,y_0^{0.75}} = \frac{48}{1} \Leftrightarrow \frac{x_0}{y_0} = 16 \Leftrightarrow x_0 = 16\,y_0.$$

We plug this formula into $f(x_0, y_0) = 8$ and obtain

$$4\,(16\,y_0)^{\frac{1}{4}}\,y_0^{\frac{3}{4}} = 8 \Leftrightarrow 8\,y_0 = 8 \Leftrightarrow y_0 = 1.$$

It follows that $x_0 = 16\,y_0 = 16$.

Q4. Answer is (c). We apply the Gauss method:

$$A \quad = \quad \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{matrix} \\ \\ -(I) \\ \\ \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 0 & -2 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{matrix} -(II) \\ \\ +(II) \\ -(II) \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ :4 \\ \\ \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1/4 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix} \begin{matrix} +(III) \\ -2\,(III) \\ \\ +(III) \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 3/4 \end{pmatrix} = A^{\star}.$$

It follows that $\text{rg}(A^{\star}) = 4$ (because it has a regular 4×4 submatrix) and thus $\text{rg}(A) = \text{rg}(A^{\star}) = 4$.

Q5. Answer is (c). The following holds:

$$\begin{aligned} \det(M - \lambda I) &= \begin{pmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{pmatrix} \\ &= (2-\lambda)\,(1-\lambda)\,(-4-\lambda) + 45 + 6 \\ &\quad +5\,(1-\lambda) - 6\,(2-\lambda) + 9\,(-4-\lambda) \\ &= -\lambda^3 - \lambda^2 + 2 \\ &= -\lambda\,(\lambda - 1)\,(\lambda + 2) \end{aligned}$$

Therefore, $\det(M - \lambda I) = 0 \Leftrightarrow \lambda \in \{-2, 1, 0\}$.

Q6. Answer is (b). We have:

$$y_{k+1} = 3 + 2^{k+1} = 3 + 2 \cdot 2^k = 3 + 2\,(y_k - 3) = 3 + 2\,y_k - 6 = 2\,y_k - 3.$$

It follows that $A = 2$ and $B = -3$.

Q7. Answer is (d). We have:

$$y_{k+1} = \underbrace{-\frac{e^2}{\pi}}_{=A}\, y_k + \frac{\pi - e}{\pi}.$$

Because $A \approx -2.352$, then the solution to the first order difference equation above is oscillating and divergent.

Q8. (a). The normal form of the difference equation is

$$y_{k+1} = \frac{a}{a-2}\, y_k + \frac{4}{(a-2)^2},$$

i.e., $A = \frac{a}{a-2}$ and $B = \frac{4}{(a-2)^2}$. Clearly, $A \neq 1$. Therefore, the general solution of the difference equation is monotone and convergent if and only if $0 < A < 1$. We have:

$$0 < A < 1 \Leftrightarrow 0 < \frac{a}{a-2} < 1.$$

If $a - 2 > 0$, then $0 < a < a - 2$, and this is never satisfied.

If $a - 2 < 0$, then $0 > a > a - 2$. The second inequality is always true, hence the solution is monotone and convergent if and only if $a < 0$.