Mathematics B
Alternative Date Exam Spring Semester 2019

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05 February 2020

Points achieved

Please leave empty

Open questions	**(a)**	**(b)**	**(c)**	**(d1)**	**(d2)**	**Total**
Exercise 1	(9)	(9)	(8)	(6)	(4)	(36)
MC questions						
Exercise 2						(32)
Exercise 3						(32)
						(100)

Part I: Open questions (36 points)

General instructions for open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain only a single version.

(vi) Temporary computations or sketches must be reported on separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (36 points)

(a) (9 points)

A production site manufactures two goods. p_1 denotes the price per unit of good 1 and p_2 the price per unit of good 2, respectively. The demand for each good depends on its price:

$$\begin{aligned} \text{Demand for good 1:} \quad x &= 100 - 5\,p_1, \\ \text{Demand for good 2:} \quad y &= 200 - 4\,p_2. \end{aligned}$$

The manufacturing costs are

$$K(x, y) = x^2 + x\,y + y^2.$$

The production manager looks for the prices p_1^* and p_2^*, for which the net profit, i.e. revenue minus costs, is maximal. Assume that the demand for each good corresponds to the quantity produced.

Prove that the optimal prices are actually delivering a maximum and not a minimum and compute the corresponding net profit.

Exercise 1

(a) (additional space for the solution)

Exercise 1

(b) (9 points)

A manufacturer for textiles imports his wool from two different suppliers. He gets x tons of woold from the first supplier and y tons from the second supplier. His purchase costs are:

$$\begin{aligned} &\text{Costs for wool from supplier 1:} \quad K_1(x) = \frac{3}{4}\,x^2 + 400, \\ &\text{Costs for wool from supplier 2:} \quad K_2(y) = \frac{11}{4}\,y^2 + 10\,y + 300. \end{aligned}$$

The manufacturer aims to produce at minimal costs a certain amount of shirts, for which he requires 100 tons of wool.

Find the amounts of wool x^* and y^* from supplier 1 and 2 for which the costs of the required demand of wool is minimal.

Prove that you actually found a minimum and compute the total costs for the wool.

Exercise 1

(b) (additional space for the solution)

Exercise 1

(c) (8 points)

For the production of the four products P_1, P_2, P_3 and P_4, four machines M_1, M_2, M_3 and M_4 are necessary.
The production times (per unit of each product in minutes) are represented in the following table:

	P_1	P_2	P_3	P_4
M_1	10	7	8	4
M_2	5	5	4	11
M_3	8	9	6	6
M_4	10	m	9	4

Each machine runs exactly 16 hours per day.

Describe the relationship between production times, running time of each machine per day and the units of produced goods per day in a system of equations.

For which positive $m > 0$ is it not possible to determine how many units of each product can be produced with the available capacities of the machines?

Exercise 1

(c) (additional space for the solution)

Exercise 1

(d1) (6 points)

Dora's parents have opened a personal savings account for her so that she can finance her summer vacation. At the beginning of each year, the parents deposit $5,000$ Swiss francs into the account. Dora always withdraws 60% of the existing funds at the first of July. The interests of 1.25 % p.a. are paid at the end of the year based on the weighted average of funds in the bank account throughout the year. There are no other transactions from or into the bank account.

Come up with an explicit formula for the account balance V_n at the end of year n after the parents opened the account.

Hint:
Weighted average of funds means, that interest on funds are only paid based on the fraction of the duration those funds have been in the bank account.

Exercise 1

(d2) (4 points)

Dora's parents have opened a personal savings account for her so that she can finance her summer vacation. At the beginning of each year, the parents deposit 5,000 Swiss francs into the account. Dora always withdraws 60% of the existing funds at the first of July. The interests of 1.25 % p.a. are paid at the end of the year based on the weighted average of funds in the bank account throughout the year. There are no other transactions from or into the bank account.

(*Important remark:* If you haven't been able to solve Exercise **(d1)**, please proceed with $V_n = 3440\,(1 - 0.4825^n)$.)

Assuming that Dora and her parents live infinitely long, what is the amount that V_n converges to?

At the end of which year does the account balance V_n rise above CHF 3,400 for the first time?

Exercise 1

(d2) (additional space for the solution)

Part II: Multiple-choice questions (64 points)

General instructions for multiple-choice questions:

(i) The solution must be reported on the multiple-choice solution form. Only the answers reported on the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility should be marked.

(iii) If two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions and the instructions on the multiple-choice solution form.

Exercise 2 (32 points)

Question 1 (3 points)

The function $f(x, y)$ is supposed to have a stationary point at (x_0, y_0), i.e. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.
For f to have a *local extrema* in (x_0, y_0), it is sufficient that

(a) $f_{xy}(x_0, y_0)f_{yx}(x_0, y_0) - f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) < 0$.

(b) $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 = 0$.

(c) $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0$.

(d) None of the above conditions is sufficient for a local extrema in (x_0, y_0).

Exercise 2

Question 2 (3 points)

The function

$$f(x, y) = x^2\, y$$

under the constraint

$$\varphi(x, y) = x + y = 0$$

has

(a) a maximum at the point $P = (0, 0)$.

(b) a minimum at the point $P = (0, 0)$.

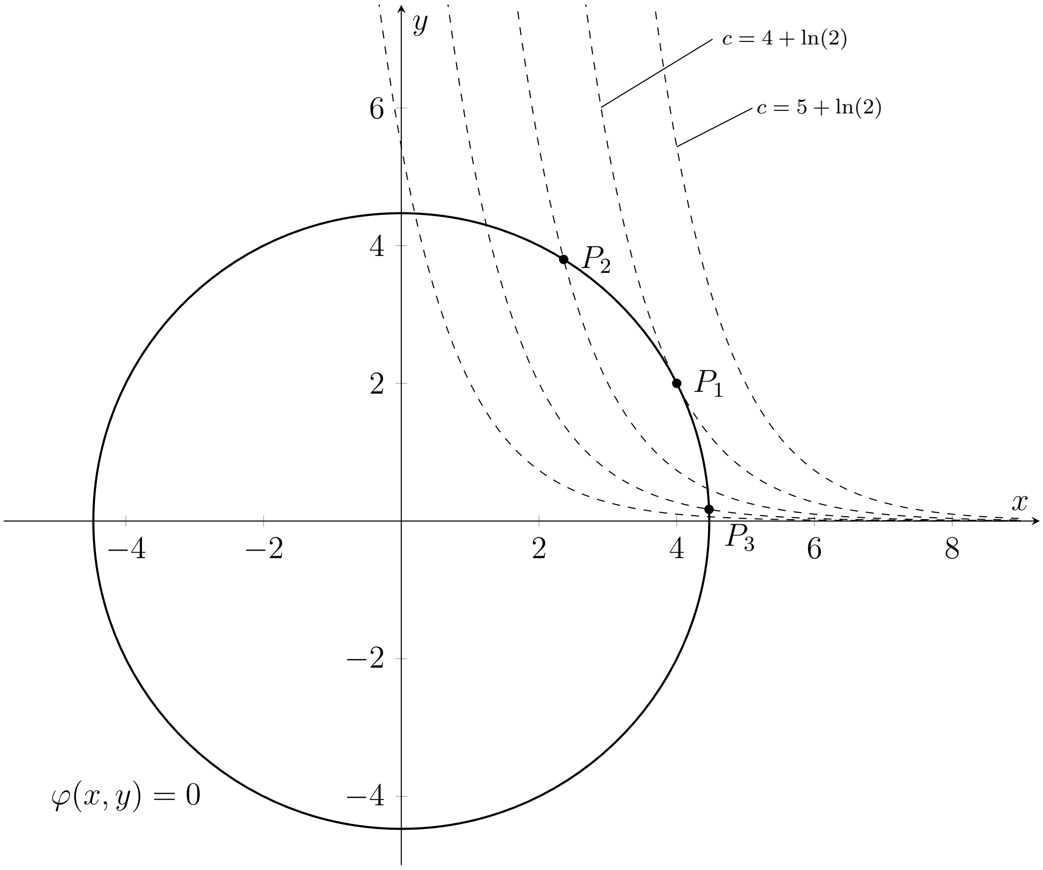
(c) a minimum at the point $P = (1, -1)$.

(d) None of the above answers are correct.

Exercise 2

Question 3 (3 points)

The following figure shows the contour lines of the function $f(x, y) = x + \ln(y)$ at different levels c and the curve $\varphi(x, y) = x^2 + y^2 - 20 = 0$.



Under the constraint $\varphi(x, y) = 0$, the function f has

(a) a maximum at the point P_1.

(b) a minimum at the point P_1.

(c) a maximum at the point P_2.

(d) a minimum at the point P_3.

Exercise 2

Question 4 (3 points)

For the functions f and g, it holds that

$$\int_b^a \mu\, f(x)\, dx = k_1 \text{ and } \int_a^b \rho\, g(x)\, dx = k_2,$$

where $\mu, \rho \in \mathbb{R} \setminus \{0\}$ and $k_1, k_2 \in \mathbb{R}$ are real numbers.

It follows:

(a) $\int_a^b (g(x) + f(x))\, dx = \mu\, k_1 + \rho\, k_2$.

(b) $\int_a^b (g(x) + f(x))\, dx = \frac{k_1}{\rho} - \frac{k_2}{\mu}$.

(c) $\int_a^b (g(x) + f(x))\, dx = \frac{k_1}{\mu} + \frac{k_2}{\rho}$.

(d) $\int_a^b (g(x) + f(x))\, dx = \frac{k_2}{\rho} - \frac{k_1}{\mu}$.

Exercise 2

Question 5 (3 points)

Given is the function

$$f(x) = \int_{x}^{4} \left(3\,t^2 - t + 1\right)\,dt.$$

Hence, it follows:

(a) $f(0) = 1$.

(b) $f'(0) = -1$.

(c) $f''(0) = -1$.

(d) $f'''(0) = 0$.

Exercise 2

Question 6 (3 points)

Let A and B two $(n \times n)$-matrices.
Hence, it follows:

(a) $(A+B)^2 = A^2 + B^2$.

(b) $(A+B)^2 = AB + B^2 + A^2 + BA$.

(c) $(A+B)^2 = A^2 + 2\,A\,B + B^2$.

(d) $(A+B)^2 = 2\,A + 2\,B$.

Exercise 2

Question 7 (4 points)

A matrix M is called *idempotent* if $M^2 = M$.
Let M be an invertible, idempotent $(n \times n)$-matrix.

Which of the following statements is *wrong*?

(a) $\det(\lambda M) = \lambda \det(M)$, for all $\lambda \in \mathbb{R}$.

(b) $\det(M) = 1$.

(c) $\det(M) = \det(M^{-1})$.

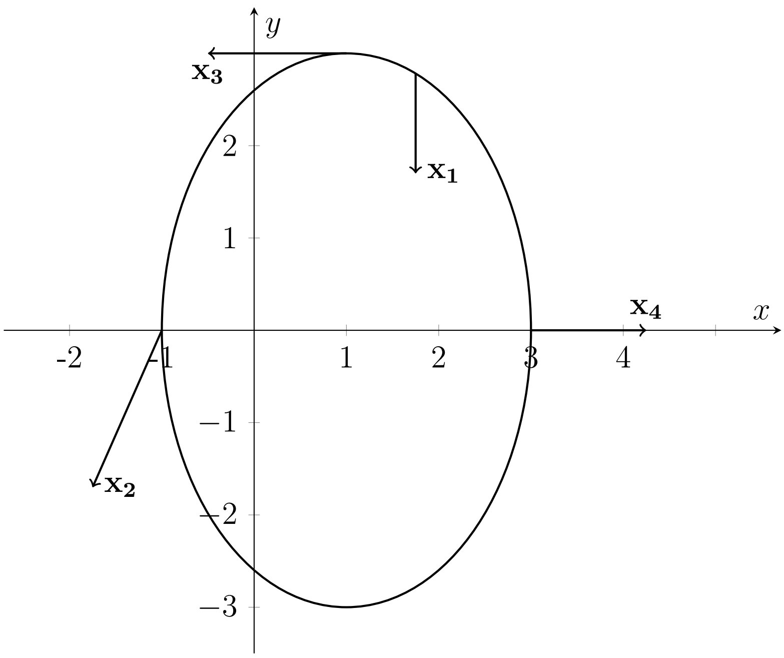
(d) $\det(M) = \det(M^T)$.

Exercise 2

Question 8 (3 points)

We consider the contour line $f(x, y) = 1$ of the function

$$f(x,y) = \frac{(x-1)^2}{4} + \frac{y^2}{9}.$$



Which of the following vectors points into the direction of the gradient of f at some point $(x_0, y_0) \in D_f$?

(a) $\mathbf{x_1}$.

(b) $\mathbf{x_2}$.

(c) $\mathbf{x_3}$.

(d) $\mathbf{x_4}$.

Exercise 2

Question 9 (3 points)

Let $A = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, ..., \mathbf{a_k}\}$ be a system of n-dimensional vectors.
Which of the following assertions is **wrong**?

(a) If $\mathbf{a_j} = \mathbf{a_i}$, for some $j \neq i$, then A is linearly dependent.

(b) If A is linearly independent, then every subsystem of A is linearly independent.

(c) If A is linearly dependent, then it holds that

$$\mathbf{a_k} = \lambda_1 \, \mathbf{a_1} + \lambda_2 \, \mathbf{a_2} + ... + \lambda_{k-1} \, \mathbf{a_{k-1}},$$

for some $\lambda_i \in \mathbb{R}$ $(i = 1, 2, 3, ..., k-1)$.

(d) If $k < n$ and A is linearly independent, then there exists an n-dimensional vector $\mathbf{a_{k+1}}$ such that $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, ..., \mathbf{a_k}, \mathbf{a_{k+1}}\}$ is linearly independent as well.

Exercise 2

Question 10 (4 points)

For the system $A\mathbf{x} = \mathbf{b}$, we have $\text{rg}(A) = 4$, where A is a (4×7)-matrix.
Hence, it follows:

(a) The system $A\mathbf{x} = \mathbf{b}$ has no solution.

(b) The system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions and the solution space has dimension 3.

(c) The system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions and the solution space has dimension 4.

(d) None of the above answers are correct.

Exercise 3 (32 points)

Question 1 (4 points)

The definite integral

$$\int_0^{\frac{\pi}{2n}} \sin(n\,x)\, e^{\cos(n\,x)}\, dx,$$

where $n \in \mathbb{N}$, has the value

(a) $\frac{1}{n}\,(e-1)$.

(b) $\frac{1}{n}\,e^{\frac{\pi}{2}} - \frac{1}{n}$.

(c) $\frac{1}{n} - 1$.

(d) None of the above values is correct.

Exercise 3

Question 2 (5 points)

Consider the function

$$f(x) = \begin{cases} \frac{1}{x^2}\ln(x) & \text{for } x \geq c \\ 0 & \text{else} \end{cases}.$$

For which value $c \in \mathbb{R}_+$ is f a density function?

(a) $c = 0$.

(b) $c = 1$.

(c) $c = \ln(2)$.

(d) $c = e$.

Exercise 3

Question 3 (4 points)

Consider the function

$$f(x, y) = 4\,x^{0.25}\,y^{0.75}.$$

At which point (x_0, y_0) of the contour line $f(x, y) = 8$ is the direction of the strongest function increase given by the vector $\mathbf{n} = \begin{pmatrix} 1 \\ 48 \end{pmatrix}$?

(a) $(x_0, y_0) = (1, 2\sqrt[3]{2})$.

(b) $(x_0, y_0) = (16, 1)$.

(c) $(x_0, y_0) = (2, 2)$.

(d) $(x_0, y_0) = (4, \sqrt[3]{4})$.

Exercise 3

Question 4 (4 points)

Given is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

A has

(a) rank 2.

(b) rank 3.

(c) rank 4.

(d) rank 5.

Exercise 3

Question 5 (4 points)

Given is the matrix

$$M = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}.$$

M has the eigenvalues

(a) $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 2$.

(b) $\lambda_1 = -1$, $\lambda_2 = 0$ and $\lambda_3 = 2$.

(c) $\lambda_1 = -2$, $\lambda_2 = 0$ and $\lambda_3 = 1$.

(d) $\lambda_1 = 0$ and $\lambda_1 = 2$.

Exercise 3

Question 6 (3 points)

For which real values A and B does the sequence $\{y_k\}_{k \in \mathbb{N}_0}$ with

$$y_k = 3 + 2^k\,, \quad k = 0, 1, 2, ...$$

represent a solution for the following difference equation?

$$y_{k+1} = A\,y_k + B\,, \quad k = 0, 1, 2, ...$$

(a) $A = 2$ and $B = 3$.

(b) $A = 2$ and $B = -3$.

(c) $A = -2$ and $B = 3$.

(d) $A = -2$ and $B = -3$.

Exercise 3

Question 7 (3 points)

The general solution of the difference equation

$$-\pi\, y_{k+1} - e^2\, y_k + (\pi - e) = 0 \quad (k = 0, 1, 2, ...)$$

is

(a) monotone and convergent.

(b) monotone and divergent.

(c) oscillating and convergent.

(d) oscillating and divergent.

Exercise 3

Question 8 (5 points)

The general solution of the linear difference equation

$$a\,(a-2)\,y_k - (a-2)^2\,y_{k+1} + 4 = 0, \quad k = 0, 1, 2, ...,$$

with $a \in \mathbb{R} \setminus \{0, 2\}$, is monotone and convergent if and only

(a) $a < 0$.

(b) $0 < a < 1$.

(c) $a > 2$

(d) There is no $a \in \mathbb{R}$ such that the solution is monotone and convergent.

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Multiple-choice answer sheet (page 1 of 2)

Exercise 2 (32 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Question 9: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
9.	☐	☐	☐	☐

Question 10: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
10.	☐	☐	☐	☐

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Multiple-choice answer sheet (page 2 of 2)

Exercise 3 (32 points)

Question 1: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (5 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (5 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐