Mathematics A
Master Solutions Exam Autumn Semester 2019

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Part I: Open questions

Exercise 1

(a1) (**2 Punkte**).
Function Q expresses how the demand $d = D(p)$ of the institutional investor affects the market price $p = P(d)$ of the stock. Mathematically speaking, Q is the composition of P and D, i.e., $Q = P \circ D$. We have:

$$Q(p) = P(D(p)) = \frac{1}{4} + \frac{1}{2}\left(4 - 2\, e^{p-1}\right) = \frac{1}{4} + 2 - e^{p-1} = \frac{9}{4} - e^{p-1}.$$

(a2) (**6 Punkte**).
In this question, we are interested in the price $p^\star \in \left[0, \frac{3}{2}\right]$ which remains unchanged after the institutional investor buys $D(p^\star)$ units of the stock. Mathematically speaking, $p^\star$ muss satisfy the so-called fixed point equation $p^\star = Q(p^\star)$. To show the existence and uniqueness of $p^\star$, we define $f(p) = Q(p) - p = \frac{9}{4} - e^{p-1} - p$ and apply Bolzano's theorem to $f(p) = 0$. Indeed:

$$p = Q(p) \Leftrightarrow f(p) = 0.$$

We have:

$$f(p) = 0 \Leftrightarrow Q(p) - p = 0 \Leftrightarrow \frac{9}{4} - e^{p-1} - p = 0.$$

Because f is continuous, according to Bolzano's Theorem $f(p) = 0$ possesses at least one solution on $\left[0, \frac{3}{2}\right]$ if the sign of f changes between 0 and $\frac{3}{2}$. We have:

$$f(0) = \frac{9}{4} - e^{0-1} - 0 = \frac{9}{4} - e^{-1} \approx 1.882121 > 0,$$

$$f\left(\frac{3}{2}\right) = \frac{9}{4} - e^{\frac{3}{2}-1} - \frac{3}{2} = \frac{3}{4} - e^{\frac{1}{2}} \approx -0.8987213 < 0.$$

Therefore, $f(p) = 0$ possesses a solution on $\left[0, \frac{3}{2}\right]$.

Next, the solution is unique if f is strictly monotone. We have:

$$f'(p) = -e^{p-1} - 1 < 0.$$

Therefore, f is strictly monotonically decreasing on $\left[0, \frac{3}{2}\right]$ and f possesses a unique solution on $\left[0, \frac{3}{2}\right]$.

In summary, we have shown that $f(p) = 0$ has a unique solution on $\left[0, \frac{3}{2}\right]$, i.e., $p = Q(p)$ has a unique solution on $\left[0, \frac{3}{2}\right]$. This implies the existence of a unique fixed point $p^\star$ of Q on $\left[0, \frac{3}{2}\right]$.

(a3) (**6 Punkte**).
In this exercise, we need to compute a second order Taylor polynom to approximate the solution of the fixed point equation $p = Q(p)$. We suggest two alternative solutions: (i) we approximate Q with the second order Taylor polynomial $P_{Q,2}$ of Q at $p_0 = 1$ and solve $p = P_{Q,2}(p)$; or (ii) we approximate f defined by $f(p) = Q(p) - p = \frac{9}{4} - e^{p-1} - p$ with the second order Taylor polynomial $P_{f,2}$ of f at $p_0 = 1$ and solve $P_{f,2}(p) = 0$.

(i) The second order Taylor polynomial $P_{Q,2}$ of Q at $p_0 = 1$ is:

$$P_{Q,2}(p) = Q(p_0) + Q'(p_0)\,(p - p_0) + \frac{1}{2}\,Q''(p_0)\,(p - p_0)^2.$$

We have:

$$\begin{aligned} Q(p_0) = Q(1) &= \frac{9}{4} - e^{1-1} = \frac{9}{4} - 1 = \frac{5}{4}, \\ Q'(p) &= -e^{p-1} \Rightarrow Q'(1) = -e^{1-1} = -1, \\ Q''(p) &= -e^{p-1} \Rightarrow Q''(1) = -e^{1-1} = -1. \end{aligned}$$

It follows that:

$$P_{Q,2}(p) = \frac{5}{4} - 1 \cdot (p-1) - \frac{1}{2} \cdot (p-1)^2 = \frac{5}{4} - p + 1 - \frac{1}{2}\,p^2 + p - \frac{1}{2} = -\frac{1}{2}\,p^2 + \frac{7}{4}.$$

Therefore,

$$p = P_{Q,2}(p) \Leftrightarrow p = -\frac{1}{2}\,p^2 + \frac{7}{4} \Leftrightarrow -\frac{1}{2}\,p^2 - p + \frac{7}{4} = 0 \Leftrightarrow -2\,p^2 - 4\,p + 7 = 0 \Leftrightarrow p = -1 \pm \sqrt{\frac{9}{2}}.$$

Since $-1 - \sqrt{\frac{9}{2}} \notin \left[0, \frac{3}{2}\right]$ we only consider $-1 + \sqrt{\frac{9}{2}} \approx 1.12132$. It follows that:

$$p^\star \approx 1.12132.$$

(ii) The second order Taylor polynomial $P_{f,2}$ of f at $p_0 = 1$ is:

$$P_{f,2}(p) = f(p_0) + f'(p_0)\,(p - p_0) + \frac{1}{2}\,f''(p_0)\,(p - p_0)^2.$$

We have:

$$\begin{aligned} f(p_0) = f(1) &= \frac{9}{4} - e^{1-1} - 1 = \frac{9}{4} - 1 - 1 = \frac{1}{4}, \\ f'(p) &= -e^{p-1} - 1 \Rightarrow f'(1) = -e^{1-1} - 1 = -2, \\ f''(p) &= -e^{p-1} \Rightarrow f''(1) = -e^{1-1} = -1. \end{aligned}$$

It follows that:

$$P_{f,2}(p) = \frac{1}{4} - 2 \cdot (p-1) - \frac{1}{2} \cdot (p-1)^2 = \frac{1}{4} - 2\,p + 2 - \frac{1}{2}\,p^2 + p - \frac{1}{2} = -\frac{1}{2}\,p^2 - p + \frac{7}{4}.$$

Therefore,

$$P_{f,2}(p) = 0 \Leftrightarrow -\frac{1}{2}\,p^2 - p + \frac{7}{4} = 0 \Leftrightarrow -2\,p^2 - 4\,p + 7 = 0 \Leftrightarrow p = -1 \pm \sqrt{\frac{9}{2}}.$$

Because $-1 - \sqrt{\frac{9}{2}} \notin \left[0, \frac{3}{2}\right]$ we only consider $-1 + \sqrt{\frac{9}{2}} \approx 1.12132$. It follows that:

$$p^\star \approx 1.12132.$$

(b) (**6 Punkte**).

The exercise gives a discrete-time description of the computational power needed to complete a given task. Originally, i.e. before the machine is upgraded, the task is completed within 15 periods. Therefore, the total computational power needed is

$$s_{15} = a_1 + \cdots + a_{15} = \sum_{n=1}^{15} a_n,$$

i.e., the 15-th partial sum of sequence $\{a_n\}_{n\in\mathbb{N}}$. Sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfies $a_1 = 10$ and $a_{n+1} = (1+2\%)\, a_n$, and thus corresponds to a geometric sequence with $a_1 = 10$ and $q = 1 + 2\% = 1.02$. It follows that:

$$s_{15} = \sum_{n=1}^{15} a_1\, q^{n-1} = a_1\, \frac{1-q^{15}}{1-q} = 10\, \frac{1-1.02^{15}}{1-1.02} \approx 172.9342.$$

After the machine is improved, the dynamics of computational power satisfies $\tilde{a}_1 = 15$ and $\tilde{a}_{n+1} = (1+3\%)\, \tilde{a}_n$, i.e., $\{\tilde{a}_n\}_{n\in\mathbb{N}}$ is again a geometric sequence with $\tilde{a}_1 = 15$ and $\tilde{q} = 1 + 3\% = 1.03$. Therefore, the total number of periods n needed by the upgraded machine to perform the given task satisfies:

$$\tilde{s}_n \overset{!}{=} s_{15} \;\Leftrightarrow\; \tilde{a}_1 + \cdots + \tilde{a}_n = \sum_{k=1}^{n} \tilde{a}_k \overset{!}{=} s_{15}\,.$$

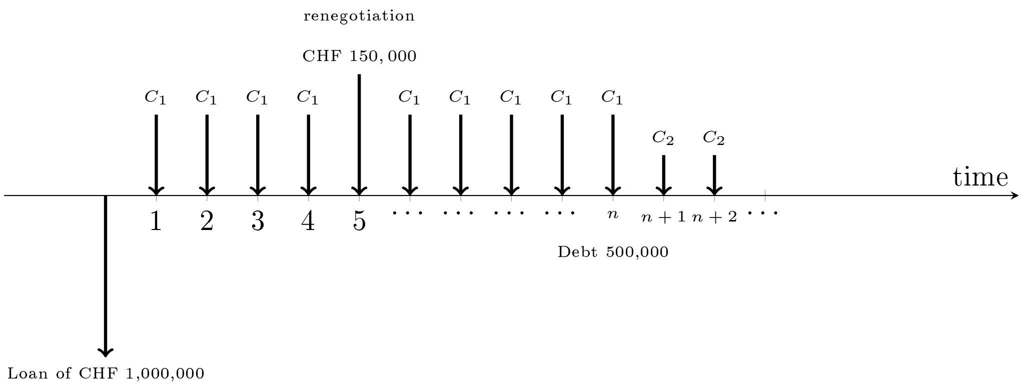
It follows that:

$$\begin{aligned}
\sum_{k=1}^{n} \tilde{a}_k = \sum_{k=1}^{n} \tilde{a}_1\, \tilde{q}^{k-1} &= \tilde{a}_1\, \frac{1-\tilde{q}^n}{1-\tilde{q}} = s_{15} \\
&\Leftrightarrow\; 15\, \frac{1-1.03^n}{1-1.03} = s_{15} \\
&\Leftrightarrow\; 1 - 1.03^n = s_{15}\, \frac{1-1.03}{15} \\
&\Leftrightarrow\; 1.03^n = 1 - s_{15}\, \frac{1-1.03}{15} = 1 - 10\, \frac{1-1.02^{15}}{1-1.02}\, \frac{1-1.03}{15} = 1.02^{15} \\
&\Leftrightarrow\; n = \frac{\ln\left(1.02^{15}\right)}{\ln(1.03)} = 15\, \frac{\ln(1.02)}{\ln(1.03)} \approx 10.0491.
\end{aligned}$$

Therefore, the upgraded machine needs slightly more than 10 periods to perform the given task. However, because the start must be postponed by 3 periods, the company will only save around $15 - 10 - 3 = 2$ periods if the upgraded machine is used instead of the original machine.

(c) (**10 points**).

(c1)



(c2) The original plan is to pay C_1 at the end of each year for 10 years in order to reduce the debt from 1,000,000 CHF to 750,000 CHF. Because the interest rate is 1%, the debt reduction in year 10 is:

$$1,000,000\,(1+1\%)^{10} - 750,000 \approx 354,622.10$$

This amount must correspond to the end value of the annuity-immediate with constant payments C_1 at the end of each year for 10 years. We have:

$$354,622.10 = \underbrace{C_1\,\frac{(1+1\%)^{10}-1}{1\%}}_{\text{final amount in year 10 of annuity-immediate}} \Leftrightarrow C_1 = \frac{354,622.10}{\frac{(1+1\%)^{10}-1}{1\%}} \approx 33,895.50 \text{ (CHF)}.$$

(c3) The debt after 5 years corresponds to the compounded value of the debt after 5 years minus the final value of the 5 yearly payments C_1 at the end of each year for 5 years, and minus the additional repayment of $150,000 - C_1$ CHF in year 5. We have:

Debt at the end of year 5

$$\begin{aligned}
&= \underbrace{1,000,000\,(1+1\%)^5}_{\text{compounded value of the debt after 5 years}} - \underbrace{C_1\,\frac{(1+1\%)^5-1}{1\%}}_{\text{value of yearly payments } C_1 \text{ after 5 years}} - \underbrace{(150,000 - C_1)}_{\text{additional payment at the end of year 5}} \\
&\approx 1,051,010 - 33,895.50\,\frac{(1+1\%)^5-1}{1\%} - 150,000 + 33,895.50 \\
&\approx 1,051,010 - 172,901.10 - 150,000 + 33,895.50 \\
&= 762,004.4 \text{ (CHF)}.
\end{aligned}$$

(c4) After year 5, the interest rate has been lowered from 1% to 0.5%. Nevertheless, the individual continues to pay C_1 at the end of each year until the debt reaches 500,000 CHF in n years. We have:

$$\begin{aligned}
\underbrace{500,000}_{\text{desired debt level after } n \text{ years}} &= \underbrace{762,004.4\,(1+0.5\%)^n}_{\text{compounded debt level after } n \text{ years}} - \underbrace{C_1\,\frac{(1+0.5\%)^n-1}{0.5\%}}_{\text{value of yearly payments } C_1 \text{ after } n \text{ years}} \\
&= \left(762,004.4 - \frac{C_1}{0.5\%}\right)(1+0.5\%)^n + \frac{C_1}{0.5\%}.
\end{aligned}$$

It follows that:

$$(1+0.5\%)^n = \frac{500,000 - \frac{C_1}{0.5\%}}{762,004.4 - \frac{C_1}{0.5\%}} \Leftrightarrow n = \frac{\ln\left(\frac{500,000-\frac{C_1}{0.5\%}}{762,004.4-\frac{C_1}{0.5\%}}\right)}{\ln(1+0.5\%)} \approx 8.54.$$

I.e., the individual will reach her desired debt level after 9 years from the renegotiation of the debt.

Please note that after 9 years the debt level is lower than 500,000 CHF. Indeed,

$$\underbrace{762,004.4\,(1+0.5\%)^9}_{\text{compounded debt level after 9 years}} - \underbrace{\qquad}_{\text{value of yearly payments } C_1 \text{ after 9 years}} = 485,756.10 \text{ (CHF)}.$$

However, we accept both the exact answer 8.54 years as well as the approximated answer 9 years. In exercise (c5) the debt level is assumed to 500,000 CHF.

(c5) After the desired debt level of 500,000 CHF has been reached, the individual pays C_2 CHF at the end of each year to maintain the debt constant at 500,000 CHF. This means that C_2 corresponds to the annual interest charged on the debt, i.e.,

$$C_2 = 0.5\% \cdot 500,000 = 2,500 \text{ (CHF)}.$$

(d) (**10 points**).

(d1) Profits correspond to revenues minus costs. In case the selling price is $p > 20$, revenues are given by:

$$q_d(p) \cdot p$$

while costs are given by

$$q_d(p) \cdot 20$$

as the production cost per tablet is 20 (USD). It follows that:

$$\text{profits} = P(p) = \text{revenues}-\text{costs} = q_d(p)\cdot p - q_d(p)\cdot 20 = q_d(p)\,(p-20) = \frac{(15,840-30\,p)\,(p-20)}{p+50}.$$

(d2) The elasticity of profits with respect to the selling price is:

$$\varepsilon_P(p) = p\,\frac{P'(p)}{P(p)}.$$

We have:

$$\begin{aligned} P'(p) &= \frac{(-30\,(p-20) + (15,840-30\,p))\,(p+50) - (15,840-30\,p)\,(p-20)}{(p+50)^2} \\ &= \frac{-30\,(p-20)\,(p+50) + (15,840-30\,p)\,(p+50-p+20)}{(p+50)^2} \\ &= \frac{-30\,(p-20)\,(p+50) + 70\,(15,840-30\,p)}{(p+50)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\varepsilon_P(p) &= p\,\frac{-30\,(p-20)\,(p+50)+70\,(15,840-30\,p)}{(p+50)^2}\,\frac{p+50}{(15,840-30\,p)\,(p-20)} \\
&= \frac{-30\,p}{15,840-30\,p}+\frac{70\,p}{(p+50)\,(p-20)}.
\end{aligned}$$

Alternatively, we could compute the rate of change of $P(p)$ with the help of the logarithmic derivative and multiply the result with p to get the elasticity $\varepsilon_P(p)$:

$$\begin{aligned}
\frac{P'(p)}{P(p)} = [\ln(P(p))]' &= \left[\ln\left(\frac{(15,840-30\,p)\,(p-20)}{p+50}\right)\right]' \\
&= [\ln(15,840-30\,p)+\ln(p-20)-\ln(p+50)]' \\
&= \frac{-30}{15,840-30\,p}+\frac{1}{p-20}-\frac{1}{p+50} \\
&= \frac{1}{p-528}+\frac{1}{p-20}-\frac{1}{p+50}.
\end{aligned}$$

Therefore,

$$\varepsilon_P(p) = p\,\frac{P'(p)}{P(p)} = \frac{p}{p-528}+\frac{p}{p-20}-\frac{p}{p+50}.$$

(d3) The relative change of profits when the price goes from 100 to 105 is:

$$\frac{P(105)-P(100)}{P(100)}.$$

The following approximation holds:

$$\frac{P(105)-P(100)}{P(100)} \approx \varepsilon_P(100)\,\frac{105-100}{100} = \varepsilon_P(100)\cdot 5\%.$$

We have:

$$\varepsilon_P(100) = \frac{-30\cdot 100}{15,840-30\cdot 100}+\frac{70\cdot 100}{(100+50)\,(100-20)} \approx 0.35.$$

Therefore,

$$\frac{P(105)-P(100)}{P(100)} \approx 0.35\cdot 5\% = 1.75\%.$$

Part II: Multiple-choice questions

Exercise 2

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☒
2.	☐	☐	☒	☐
3.	☐	☐	☐	☒
4.	☐	☐	☒	☐
5.	☒	☐	☐	☐
6.	☐	☒	☐	☐
7.	☒	☐	☐	☐
8.	☐	☐	☒	☐
9.	☐	☐	☐	☒
10.	☐	☒	☐	☐

1. Answer is (d). The following truth tables show only the proposition in (d) is true if and only if A differs from B:

	A	T	T	F	F
	B	T	F	T	F
	$A \vee B$	T	T	T	F
	$A \wedge B$	T	F	F	F
	$A \Rightarrow B$	T	F	T	T
	$\neg(A \wedge B)$	F	T	T	T
(a)	$A \vee B$	T	T	T	F
(b)	$A \wedge B$	T	F	F	F
(c)	$\neg(A \Rightarrow B)$	F	T	F	F
(d)	$(A \vee B) \wedge (\neg(A \wedge B))$	F	T	T	F

2. Answer is (c). If John didn't join the party, then A is false. Therefore, the implication $A \Rightarrow B$ is true, independently from whether B is true or false. By contrast, $A \wedge B$ is false, $A \vee B$ could be true or false (depending on B), and $A \Leftrightarrow B$ could be true or false (depending on B).

3. Answer is (d). (a) and (c) are generally wrong, as can be seen by setting $a_n = \frac{1}{n}$. In this case $\{a_n\}_{n\in\mathbb{N}}$ is bounded, monotone, and convergent. However, because $b_n = \frac{1}{a_n} = n$, then $\{b_n\}_{n\in\mathbb{N}}$ is neither bounded, nor convergent. Furthermore, if $\{a_n\}_{n\in\mathbb{N}}$ is monotone and changes sign, then $\{b_n\}_{n\in\mathbb{N}}$ is not monotone. E.g. the sequence $a_n = \frac{1}{n} - 0.9$ is bounded, monotone, and convergent, while $b_n = \frac{1}{a_n}$ is bounded and convergent but not monotone. Therefore answer (d) is true.

4. Answer is (c). $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are both geometric sequences with parameters q and $\frac{q}{2}$, respectively. We have:

$$\sum_{k=1}^{\infty} a_k = \frac{a_1}{1-q}$$

and

$$\sum_{k=1}^{\infty} b_k = \frac{b_1}{1-\frac{q}{2}}.$$

This latter condition is satisfied when $a_1 = 2\,b_1$ and $1 - q = 2 - q$ It follows that:

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k \Leftrightarrow \frac{a_1}{1-q} = \frac{b_1}{1-\frac{q}{2}} = \frac{2\,b_1}{2-q}.$$

If $b_1 = 2\,a_1$, then:

$$\frac{a_1}{1-q} = \frac{4\,a_1}{2-q} \Leftrightarrow \begin{cases} a_1 = 0, q \in (-1,1) \\ a_1 \neq 0, q = \frac{2}{3} \end{cases}.$$

5. Answer is (a). Mark repays the loan before Lucie. Indeed, Mark will be able to repay the loan within a finite amount of time, because his annual payments are always larger than his annual interests:

$$1,000,000 \cdot 1\% = 10,000 < 10,500.$$

By contrast, Lucie will never be able to repay the loan, because her annual payments do not even allow to repay her annual interests:

$$800,000 \cdot 1.2\% = 9,600 > 9,500.$$

6. Answer is (b). Because h is only defined when g is defined, then $D_h \subseteq D_g$.

7. Answer is (a). The elasticity corresponds to:

$$\varepsilon_f(x) = x\,\frac{f'(x)}{f(x)} = x\,\rho_f(x)$$

where $\rho_f(x)$ is the rate of change. It follows that:

$$\rho_f(x) = \frac{\varepsilon_f(x)}{x}.$$

Because $\varepsilon_f(x) = 2$, then

$$\rho_f(x) = \frac{2}{x}.$$

Therefore, because $D_f \subseteq \mathbb{R}_{++}$, then ρ_f is strictly decreasing.

8. Answer is (c). The elasticity corresponds to:

$$\varepsilon_f(x) = x\,\frac{f'(x)}{f(x)}.$$

Therefore, if $x_0 \neq 0$, $f(x) > 0$ for all x, and $\varepsilon_f(x_0) = 0$, then

$$f'(x_0) = 0.$$

It follows that x_0 is a stationary point. Because f is strictly concave then x_0 is local maximum (it is even a global maximum, but every global maximum then is also a local maximum).

9. Answer is (d). The reminder term is generally neither decreasing or increasing with the order of the Taylor polynomial. All cases are possible.

10. Answer is (b). Because f is homogeneous of degree 2, then according to Euler's relation:

$$\varepsilon_{f,x}(x,y) + \varepsilon_{f,y}(x,y) = 2.$$

It follows that:

$$\varepsilon_{f,x}(x,y) = 2 - \varepsilon_{f,y}(x,y) = 2 - (e^{x+y} + 1) = 1 - e^{x+y}.$$

Exercise 3

	(a)	(b)	(c)	(d)
1.	☒	☐	☐	☐
2.	☐	☒	☐	☐
3.	☐	☐	☒	☐
4.	☐	☐	☐	☒
5.	☐	☐	☒	☐
6.	☐	☒	☐	☐
7.	☐	☐	☒	☐
8.	☐	☒	☐	☐

1. Answer is (a). We have:

$$\begin{aligned}
\lim_{x\to 1+}\left(\frac{1}{x-1}-\frac{1}{\ln(x)}\right) &= \lim_{x\to 1+}\frac{\ln(x)-(x-1)}{(x-1)\,\ln(x)} \\
&\overset{(\star)}{=} \lim_{x\to 1+}\frac{\frac{1}{x}-1}{\ln(x)+(x-1)\,\frac{1}{x}} \\
&= \lim_{x\to 1+}\frac{1-x}{x\,\ln(x)+(x-1)} \\
&\overset{(\star)}{=} \lim_{x\to 1+}\frac{-1}{\ln(x)+x\,\frac{1}{x}+1} \\
&= \lim_{x\to 1+}\frac{-1}{\ln(x)+2} \\
&= -\frac{1}{2}.
\end{aligned}$$

In $(\star)$ we apply L'Hôpital rule.

2. Answer is (b). The following applies:

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{\ln(x^x)} = \lim_{x\to 0+} e^{x\,\ln(x)} = e^{\lim_{x\to 0+}(x\,\ln(x))}.$$

Because:

$$\lim_{x\to 0+}(x\,\ln(x)) = \lim_{x\to 0+}\frac{\ln(x)}{\frac{1}{x}} \overset{\star}{=} \lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+}(-x) = 0$$

then

$$\lim_{x\to 0+} x^x = e^{\lim_{x\to 0+}(x\,\ln(x))} = e^0 = 1.$$

In $(\star)$ we apply L'Hôpital rule.

3. Answer is (c). Let a_n be the paper's thickness after it is folded n times. We have:

$$a_0 = 0.5 \text{ (millimeter)} = 0.5\,10^{-6} \text{ (kilometers)}$$

and

$$a_n = 2\,a_{n-1}.$$

Therefore, $\{a_n\}_{n\in\mathbb{N}}$ is a geometric sequence with $a_0 = 0.5\,10^{-6}$ and $q = 2$. It follows that:

$$a_n = a_0\,q^n$$

and

$$a_n \geq 400,000 = 4 \cdot 10^5 \Leftrightarrow 0.5\, 10^{-6}\, 2^n \geq 4 \cdot 10^5.$$

It follows that:

$$2^n \geq \frac{4 \cdot 10^5}{0.5 \cdot 10^{-6}} = 8 \cdot 10^{11} \Leftrightarrow n \geq \frac{\ln(8 \cdot 10^{11})}{\ln(2)} \approx 39.54.$$

Therefore, the paper should be folded at least 40 times.

4. Answer is (d). The internal rate of return r implies a project's net present value of zero. We have:

$$-10,000 + \frac{5,000}{1+r} + \frac{10,000}{(1+r)^2} = 0.$$

Multiplying this latter equation by $\frac{(1+r)^2}{5,000}$ leads to:

$$-2\,(1+r)^2 + (1+r) + 2 = 0 \Leftrightarrow 1+r = \frac{-1 \pm \sqrt{1^2+16}}{-4} = \frac{-1 \pm \sqrt{17}}{-4}.$$

Therefore, $1+r \approx 1.3$, i.e., $r \approx 30\%$.

5. Answer is (c). We have:

$$(x,y) \in D_f \Leftrightarrow x^2+y^2-1 > 0 \text{ and } 4-x^2-y^2 > 0.$$

The first condition corresponds to $x^2+y^2 > 1^2$, i.e., all points (x,y) outside the circle with centre $(0,0)$ and radius 1. The second condition corresponds to $x^2+y^2 < 2^2$, i.e., all points (x,y) inside the circle with center $(0,0)$ and radius 2.

6. Answer is (b). The implicit function theorem implies that the slope to the tangent line to the contour line of f at the point $(x_0,y_0) = (1,1)$ corresponds to:

$$-\frac{f_x(x_0,y_0)}{f_y(x_0,y_0)} = -\frac{10\,\alpha\, x_0^{\alpha-1}\, y^{1-\alpha}}{10\,(1-\alpha)\, x_0^{\alpha}\, y_0^{-\alpha}} \overset{(x_0,y_0)=(1,1)}{=} -\frac{\alpha}{1-\alpha}.$$

It follows that:

$$-\frac{\alpha}{1-\alpha} = -0.5 \Leftrightarrow \alpha = 0.5\,(1-\alpha) \Leftrightarrow 1.5\,\alpha = 0.5 \Leftrightarrow \alpha = \frac{1}{3}.$$

7. Answer is (c). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\, x, \lambda\, y) &= \ln\left((\lambda\, x)^3\, \sqrt[3]{(\lambda\, y)^4} + \sqrt[6]{(\lambda\, x)^{11}\,(\lambda\, y)^{15}}\right) - \frac{13}{3}\ln(\lambda\, x) \\
&= \ln\left(\lambda^{\frac{13}{3}}\, x^3\, \sqrt[3]{y^4} + \lambda^{\frac{13}{3}}\, \sqrt[6]{x^{11}\, y^{15}}\right) - \frac{13}{3}\ln(x) - \frac{13}{3}\ln(\lambda) \\
&= \ln\left(\lambda^{\frac{13}{3}}\right) + \ln\left(x^3\, \sqrt[3]{y^4} + \sqrt[6]{x^{11}\, y^{15}}\right) - \frac{13}{3}\ln(x) - \frac{13}{3}\ln(\lambda) \\
&= \frac{13}{3}\ln(\lambda) + \ln\left(x^3\, \sqrt[3]{y^4} + \sqrt[6]{x^{11}\, y^{15}}\right) - \frac{13}{3}\ln(x) - \frac{13}{3}\ln(\lambda) \\
&= \ln\left(x^3\, \sqrt[3]{y^4} + \sqrt[6]{x^{11}\, y^{15}}\right) - \frac{13}{3}\ln(x) \\
&= f(x,y) \\
&= \lambda^0\, f(x,y).
\end{aligned}$$

It follows that f is homogeneous of degree 0.

8. Answer is (b). For $\lambda > 0$ we have:

$$\begin{aligned} h(\lambda\, x, \lambda\, y) &= f(g(\lambda\, x, \lambda\, y)) \\ &= f(\lambda^{\kappa}\, g(x,y)) \\ &= (\lambda^{\kappa}\, g(x,y))^{\alpha} \\ &= \lambda^{\alpha\,\kappa}\, g(x,y)^{\alpha} \\ &= \lambda^{\alpha\,\kappa}\, f(g(x,y)) \\ &= \lambda^{\alpha\,\kappa}\, h(x,y). \end{aligned}$$

Therefore, f is homogeneous of degree $\alpha\,\kappa$ and:

(a) $\alpha\,\kappa = 0.5 = 0.5 \cdot 1$ if $\alpha = 0.5$ and $\kappa = 1$,

(b) $\alpha\,\kappa = 2 = 0.5 \cdot 4$ if $\alpha = 0.5$ and $\kappa = 4$,

(c) $\alpha\,\kappa = 3.5 = 0.5 \cdot 7$ if $\alpha = 0.5$ and $\kappa = 7$,

(d) $\alpha\,\kappa = 5.5 = 0.5 \cdot 11$ if $\alpha = 0.5$ and $\kappa = 11$.