Mathematics A
Exam Autumn Semester 2019

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Points achieved

Please leave empty							
	Open questions	(a1|a2)	(a3)	(b)	(c)	(d)	Total
	Exercise 1	(8)	(6)	(6)	(10)	(10)	(40)
	MC questions						
	Exercise 2						(32)
	Exercise 3						(28)
							(100)

Part I: Open questions (40 points)

General instructions for open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain only a single version.

(vi) Temporary computations or sketches must be reported on separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (40 points)

An institutional investor intends to buy many units of a given stock. However, because the market price p of the stock also depends on its market demand, the investor wants to first understand how the price p is affected by her own demand for the stock. The investor's demand for the stock corresponds to $d = D(p) = 4 - 2\,e^{p-1}$. The price of the stock depends on the investor's demand d according to $P(d) = \frac{1}{4} + \frac{1}{2}\,d$.

(a1) (2 points) Combine functions P and D to obtain function $Q = P \circ D$,

$$Q : \mathbb{R}_+ \to \mathbb{R}_+, \quad p \mapsto q = Q(p),$$

which expresses the relation between the new price q and the old price p after the investor buys $D(p)$ units of the stock. Report the expression for function Q.

(a2) (6 points) Show that a unique fixed point $p^\star \in \left[0, \frac{3}{2}\right]$ exists such that $p^\star = Q(p^\star)$, i.e., a price $p^\star$ that remains unchanged after the institutional investor buys $D(p^\star)$ units of the stock.

Exercise 1

(a2) (additional space for the solution)

Exercise 1

(a3) (6 points)

An institutional investor intends to buy many units of a given stock. However, because the market price p of the stock also depends on its market demand, the investor wants to first understand how the price p is affected by her own demand for the stock. The investor's demand for the stock corresponds to $d = D(p) = 4 - 2\,e^{p-1}$. The price of the stock depends on the investor's demand d according to $P(d) = \frac{1}{4} + \frac{1}{2}\,d$.

Function $Q = P \circ D$ possesses a unique fixed point $p^\star \in \left[0, \frac{3}{2}\right]$ with $p^\star = Q(p^\star)$.

Approximate $p^\star$ using a second order Taylor polynomial at $p_0 = 1$.

Exercise 1

(a3) (additional space for the solution)

Exercise 1

(b) (6 points)

A computer has computational power a_n during period n for $n = 1, 2, \ldots$ Computational power increases by 2% in each period. Assuming $a_1 = 10$, a given computationally intensive task requires 15 periods to be completed (i.e., the total computational power over the first 15 periods is needed to perform the task).

Instead of immediately starting with the task, the IT department of the company decides to first improve the computer's computational power. Consequently the task will only be started in 3 periods, but the improved machine now has an initial power of 15, which afterwards increases by 3% in each period.

How much time does the company save if the task is postponed and performed with the improved machine?

Exercise 1

(b) (additional space for the solution)

Exercise 1

(c) (10 points)

An individual borrows 1,000,000 CHF to buy her house. The interest rate corresponds to $i = 1\%$. Initially, the individual agrees to pay C_1 CHF at the end of each year for 10 years in order to reduce her debt to 750,000 CHF at the end of the tenth year. However, at the end of the fifth year, the individual pays a total amount of 150,000 CHF (including C_1) and renegotiates the conditions with the bank, such that the new interest rate is lower at 0.5%. However, the individual continues to pay C_1 CHF at the end of each year until her debt reaches 500,000 CHF. Afterwards the individual pays C_2 CHF at the end of each year such that his debt stays constant at 500,000 CHF, because this is fiscally more convenient.

(c1) Add all events and cashflows to the timeline.

(c2) Compute C_1.

(c3) What is the debt after the individual pays 150,000 CHF at the end of the fifth year?

(c4) How long does it take after the fifth year to reach the desired debt level of 500,000 CHF?

(c5) Compute C_2.

Exercise 1

(c) (additional space for the solution)

Exercise 1

(d) (10 points)

A supplier of tablets estimates the demand function q_d for its newest tablet:

$$q_d(p) = \frac{15,840 - 30\,p}{p + 50},$$

where $p > 20$ is the selling price in U.S. dollars per tablet. The production and distribution cost is 20 U.S. dollars per tablet.

(d1) Derive the supplier's profit function for selling prices $p > 20$.

(d2) Derive the function that describes the elasticity of profits with respect to the selling price.

(d3) Apply the elasticity function to approximate the relative change of profits if the initial selling price of $p_0 = 100$ U.S. dollar is increased by 5 U.S. dollars.

Exercise 1

(d) (additional space for the solution)

Exercise 1

(d) (additional space for the solution)

Part II: Multiple-choice questions (60 points)

General instructions for multiple-choice questions:

(i) The solution must be reported on the multiple-choice solution form. Only the answers reported on the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes and will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility should be marked.

(iii) If two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions and the instructions on the multiple-choice solution form.

Exercise 2 (32 points)

Question 1 (4 points)

Which of the following propositions is true if and only A and B differs (one is true and the other is false)?

(a) $A \vee B$

(b) $A \wedge B$

(c) $\neg(A \Rightarrow B)$

(d) $(A \vee B) \wedge (\neg(A \wedge B))$

Exercise 2

Question 2 (3 points)

Let $A =$“Jane and John join the party” and $B =$“Mark joins the party”.

We know that John didn’t join the party. Nothing is known about Jane and Mark.

Which of the following propositions is hence true?

(a) $A \wedge B$

(b) $A \vee B$

(c) $A \Rightarrow B$

(d) $A \Leftrightarrow B$

Exercise 2

Question 3 (3 points)

The sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded, monotone and convergent. Moreover, $a_n \neq 0$ for all n. Let $\{b_n\}_{n\in\mathbb{N}}$ be the sequence defined by $b_n = \frac{1}{a_n}$.

It follows that:

(a) $\{b_n\}_{n\in\mathbb{N}}$ is bounded.

(b) $\{b_n\}_{n\in\mathbb{N}}$ is monotone.

(c) $\{b_n\}_{n\in\mathbb{N}}$ is convergent.

(d) None of the above properties hold.

Exercise 2

Question 4 (4 points)

Let $\{a_n\}_{n\in\mathbb{N}}$ be a geometric sequence with $\frac{a_{n+1}}{a_n} = q \in (0,1)$. Let $\{b_n\}_{n\in\mathbb{N}}$ be another geometric sequence with $\frac{b_{n+1}}{b_n} = \frac{q}{2}$.
We have: $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k$. It follows that:

(a) $b_1 = 2\,a_1$ if $q = \frac{1}{3}$.

(b) $b_1 = 2\,a_1$ if $q = \frac{1}{2}$.

(c) $b_1 = 2\,a_1$ if $q = \frac{2}{3}$.

(d) The condition $b_1 = 2\,a_1$ is never satisfied.

Exercise 2

Question 5 (3 points)

The same day Mark takes a loan of $P_1 = 1,000,000$ CHF with an interest rate $i_1 = 1\%$, Lucie takes a loan of $P_2 = 800,000$ CHF with an interest rate $i_2 = 1.2\%$. Mark repays the loan with constant payments of 10,500 CHF at the end of each year, while Lucie pays 9,500 CHF at the end of each year.

Which of the following statements is correct?

(a) Mark repays the loan before Lucie.

(b) Lucie repays the loan before Mark.

(c) Lucie and Mark repay the loan at the same time.

(d) None of the above statements are correct.

Exercise 2

Question 6 (3 points)

Let f and g be functions of one real variable with domain D_f and D_g, respectively.

Let $h = f \circ g$. Which of the following statements about the domain D_h of h is correct?

(a) $D_h = D_f \cup D_g$

(b) $D_h \subseteq D_g$

(c) $D_h \subseteq D_f$

(d) $D_h = D_f \cap D_g$

Exercise 2

Question 7 (3 points)

For a given function f with domain $D_f \subseteq \mathbb{R}_{++}$ the elasticity ε_f is constant and equal to 2. Which of the following statements on the rate of change ρ_f of f is correct?

(a) ρ_f is strictly decreasing.

(b) ρ_f is strictly increasing.

(c) ρ_f is constant.

(d) ρ_f is non-monotonic.

Exercise 2

Question 8 (4 points)

A differentiable function $f : D_f \to \mathbb{R}$ is strictly concave and satisfies $f(x) > 0$ for all $x \in D_f$. Moreover, $x_0 \in D_f$, $x_0 \neq 0$ exists such that the elasticity of f satisfies $\varepsilon_f(x_0) = 0$.

Which of the following statements is correct:

(a) At x_0, f is elastic.

(b) At x_0, f possesses a local minimum.

(c) At x_0, f possesses a local maximum.

(d) None of the above answers are correct.

Exercise 2

Question 9 (3 points)

Let f be function of a real variable that is at least n times differentiable. Let P_k be the k-th order Taylor polynomial of f at x_0 for $k = 1, \dots, n-1$ and R_k the corresponding k-th order remainder term.

Which of the following statements is correct?

(a) For all $x \in D_f$ and $k = 2, \dots, n-1$, $R_k(x) < R_{k-1}(x)$.

(b) For all $x \in D_f$ and $k = 2, \dots, n-1$, $R_k(x) > R_{k-1}(x)$.

(c) For all $x \in D_f$ and $k = 2, \dots, n-1$, $R_k(x) = R_{k-1}(x)$.

(d) None of the statements above are correct.

Exercise 2

Question 10 (2 points)

A function of two variables f is homogeneous of degree 2 and its partial elasticity satisfies

$$\varepsilon_{f,y}(x,y) = e^{x+y} + 1.$$

It follows that:

(a) $\varepsilon_{f,x}(x,y) = 1 + e^{x+y}$.

(b) $\varepsilon_{f,x}(x,y) = 1 - e^{x+y}$.

(c) $\varepsilon_{f,x}(x,y) = e^{x+y}$.

(d) $\varepsilon_{f,x}(x,y) = -e^{x+y}$.

Exercise 3 (28 points)

Question 1 (4 points)

The right-sided limit

$$\lim_{x \to 1+} \left(\frac{1}{x-1} - \frac{1}{\ln(x)} \right)$$

is equal to:

(a) $-\frac{1}{2}$.

(b) 0.

(c) $\frac{1}{2}$.

(d) ∞.

Exercise 3

Question 2 (4 points)

The right-sided limit

$$\lim_{x \to 0+} x^x$$

is equal to:

(a) 0.

(b) 1.

(c) e.

(d) e^e.

Exercise 3

Question 3 (4 points)

A sheet of paper is 0.5 millimeter thick. Suppose you fold it in half to reach a total thickness of 1 millimeter. You then fold it again in half, and then again in half as long as its total thickness reaches 400,000 kilometers (approximatively the distance earth-moon).

How often should you fold the paper in half?

(a) Approximately 20 times.

(b) Approximately 30 times.

(c) Approximately 40 times.

(d) Approximately 50 times.

Exercise 3

Question 4 (4 points)

A project requires an initial investment of 10,000 CHF. After 1 year it returns 5,000 CHF and after 2 years it returns 10,000 CHF.

The internal rate of the project is approximately

(a) 5%.

(b) 10%.

(c) 20%.

(d) 30%.

Exercise 3

Question 5 (3 points)

Consider the function $f : D_f \to \mathbb{R}, \quad (x, y) \mapsto z = f(x, y) = \frac{\ln(x^2+y^2-1)}{\sqrt{4-x^2-y^2}}$.

(a) The domain of f is the inside (without contour) of the circle with centre $(0, 0)$ and radius 1.

(b) The domain of f is the inside (without contour) of the circle with centre $(0, 0)$ and radius 2.

(c) The domain of f is the inside (without contour) of the circle with centre $(0, 0)$ and radius 2, excluding the inside (including contour) of the circle with centre $(0, 0)$ and radius 1.

(d) The domain of f is empty.

Exercise 3

Question 6 (3 points)

Let f be a function of two real variables defined by:

$$f\,:\,\mathbb{R}^2_{++} \to \mathbb{R}, \quad (x,y) \mapsto z = f(x,y) = 10\,x^{\alpha}\,y^{1-\alpha},$$

where $\alpha \in (0,1)$.

The slope of the tangent line to the contour line of f at the point $(x_0, y_0) = (1,1)$ equals -0.5.

It follows that:

(a) $\alpha = \frac{1}{6}$.

(b) $\alpha = \frac{1}{3}$.

(c) $\alpha = \frac{1}{2}$.

(d) $\alpha = \frac{2}{3}$.

Exercise 3

Question 7 (3 points)

Given is the function

$$f(x,y) = \ln\left(x^3\,\sqrt[3]{y^4} + \sqrt[6]{x^{11}\,y^{15}}\right) - \frac{13}{3}\ln(x) \qquad \text{for } x > 0,\ y > 0.$$

Which of the following statements is correct?

(a) f is homogeneous of degree 5.

(b) f is linear homogeneous.

(c) f is homogeneous of degree 0.

(d) f is not homogeneous.

Exercise 3

Question 8 (3 points)

Let f be a function of a real variable defined by $f(x) = x^{\alpha}$ for $\alpha > 0$ and g be a function of two real variables, which is strictly positive and homogeneous of degree κ.

We define function h as $h(x, y) = f(g(x, y))$.

Which of the following statements is correct?

(a) h is homogeneous of degree 1 if $\alpha = 0.5$ and $\kappa = 1$.

(b) h is homogeneous of degree 2 if $\alpha = 0.5$ and $\kappa = 4$.

(c) h is homogeneous of degree 3 if $\alpha = 0.5$ and $\kappa = 7$.

(d) h is homogeneous of degree 5 if $\alpha = 0.5$ and $\kappa = 11$.

Exams Assessment Level: Autumn Semester 2019

1'202 Mathematics A

Multiple-choice answer sheet (page 1 of 2)

Exercise 2 (32 points)

Question 1: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Question 9: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
9.	☐	☐	☐	☐

Question 10: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
10.	☐	☐	☐	☐

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Multiple-choice answer sheet (page 2 of 2)

Exercise 3 (28 points)

Question 1: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐