Mathematics B
Master Solution Spring Semester 2018

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Part I: Open Questions

Open questions require more than just applying a set of formulas or equations. Therefore, simply searching for the right formula is very likely not a successful strategy. By contrast, when reading the problem one should first identify key words that point to the most useful mathematical framework, and then, in a second stage, verify if the information provided allows to apply specific mathematical results. For example, if the text refers to "maximization" or "minimization", then very likely one has to apply mathematical optimisation and thus verify if this refers to constrained or to unconstrained optimisation. Translating the verbal description of a problem into an equivalent mathematical formulation of it is called *mathematical formalisation* and is a crucial step to make use of mathematics for solving problems. Open questions require that problems are formalised using mathematics before the adequate mathematical results and equations can be applied.

Exercise 1

(a) (8 points)

In this exercise the key words to understand which mathematical framework could be applied to solve the problem are "present value" and "continuous cash flows". The present value corresponds to the discounted sum of future cash flows (Section 8.2). However, because cash flows are generated continuously, here we need to replace the sum with an integral and use the exponential function to discount future cash flows (section 15.2). This leads us to the following mathematical equation for the present value:

$$PV(10) = \int_0^{10} B(t)\, e^{-i\,t}\, dt = \int_0^{10} (a\,t + 10)\, e^{-i\,t}\, dt.$$

Before solving the integral, we need to understand what are we looking for. Using the above formalisation, we see that the problem is about determining the parameter a such that

$$PV(10) = 1,000.$$

At this stage the plan is clear: (i) solve the integral to obtain an explicit formula for $PV(10)$, (ii) use the condition $PV(10) = 1,000$ to determine the parameter a.

(i) To solve the integral $\int_0^{10}(a\,t + 10)\, e^{-i\,t}\, dt$ we apply integration by parts with $u(t) = a\,t + 10$ and $v'(t) = e^{-i\,t}$. It follows that $u'(t) = a$ and $v(t) = -\frac{1}{i}\, e^{-i\,t}$. Therefore:

$$\begin{aligned}
\int \underbrace{(a\,t+10)}_{=u(t)}\ \underbrace{e^{-i\,t}}_{=v'(t)}\, dt &= \underbrace{(a\,t+10)}_{=u(t)} \underbrace{\left(-\frac{1}{i}\, e^{-i\,t}\right)}_{=v(t)} - \int \underbrace{a}_{=u'(t)} \underbrace{\left(-\frac{1}{i}\, e^{-i\,t}\right)}_{=v(t)} dt \\
&= -\frac{a\,t+10}{i}\, e^{-i\,t} + \frac{a}{i}\int e^{-i\,t}\, dt \\
&= -\frac{a\,t+10}{i}\, e^{-i\,t} - \frac{a}{i^2}\, e^{-i\,t} + C,
\end{aligned}$$

where $C \in \mathbb{R}$. It follows that:

$$\begin{aligned}
PV(10) &= \int_0^{10} (a\,t+10)\, e^{-i\,t}\, dt \\
&= \left[-\frac{a\,t+10}{i}\, e^{-i\,t} - \frac{a}{i^2}\, e^{-i\,t}\right]_0^{10}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{10\,a+10}{i}\,e^{-10\,i} - \frac{a}{i^2}\,e^{-10\,i} - \left(-\frac{10}{i} - \frac{a}{i^2}\right) \\
&= a\left(\frac{1}{i^2} - \frac{10}{i}\,e^{-10\,i} - \frac{1}{i^2}\,e^{-10\,i}\right) + \left(\frac{10}{i} - \frac{10}{i}\,e^{-10\,i}\right) \\
&= a\left[\frac{1}{i^2}\left(1-e^{-10\,i}\right) - \frac{10}{i}\,e^{-10\,i}\right] + \frac{10}{i}\left(1-e^{-10\,i}\right).
\end{aligned}
$$

In summary, we now have an explicit formula for the present value:

$$PV(10) = a\left[\frac{1}{i^2}\left(1-e^{-10\,i}\right) - \frac{10}{i}\,e^{-10\,i}\right] + \frac{10}{i}\left(1-e^{-10\,i}\right).$$

(ii) We now use the formula above and the condition $PV(10) = 1,000$ to compute a:

$$
\begin{aligned}
PV(10) = 1,000 \quad &\Leftrightarrow \quad a\left[\frac{1}{i^2}\left(1-e^{-10\,i}\right) - \frac{10}{i}\,e^{-10\,i}\right] + \frac{10}{i}\left(1-e^{-10\,i}\right) = 1,000 \\
&\Leftrightarrow \quad a = \frac{1,000 - \frac{10}{i}\left(1-e^{-10\,i}\right)}{\frac{1}{i^2}\left(1-e^{-10\,i}\right) - \frac{10}{i}\,e^{-10\,i}} \\
&\overset{i=0.05}{\Leftrightarrow} \quad a \approx 25.534.
\end{aligned}
$$

(b) (8 points)

In this exercise the key words to understand which mathematical framework could be applied to solve the problem are “table” and “linearly combine”. In mathematics, tables of numbers corresponds to matrices (Section 16.1) and the question whether “linearly combining” columns of a matrix allows to obtain a given vector refers to the question of whether a system of linear equations possess solutions (Chapter 18).

Specifically, the payoff

$$\begin{pmatrix} 2\,m \\ 1.0 \\ 0.5 \end{pmatrix}$$

can be achieved by linearly combining the three assets if and only if the system

$$\lambda_1 \begin{pmatrix} 1.5 \\ 1.5 \\ 1.5 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3.0 \\ 2.0 \\ 0.5 \end{pmatrix} + \lambda_3 \begin{pmatrix} m \\ 0.5 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 2\,m \\ 1.0 \\ 0.5 \end{pmatrix}$$

possess at least a solution $(\lambda_1^\star, \lambda_2^\star, \lambda_3^\star)$. Because systems of linear equations possess either a unique solution, or infinite many solutions, or no solution, then we need to identify value for m for which the system has either a unique solution or infinite many solutions. To achieve this we apply Gauss:

$$
(A, \mathbf{b}) \quad = \quad
\begin{array}{c}
\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \end{array} \\
\left(\begin{array}{ccc|c}
1.5 & 3.0 & m & 2\,m \\
1.5 & 2.0 & 0.5 & 1 \\
1.5 & 0.5 & 1.5 & 0.5
\end{array}\right)
\end{array}
\cdot \left(\tfrac{2}{3}\right)
$$

$$\begin{array}{c} \\ \to \end{array} \begin{array}{c} \begin{array}{ccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & \frac{2}{3}m & \frac{4}{3}m \\ \frac{3}{2} & 2 & \frac{1}{2} & 1 \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right) \begin{array}{l} \\ -\frac{3}{2}(I) \\ -\frac{3}{2}(I) \end{array} \end{array}$$

$$\to \begin{array}{c} \begin{array}{ccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & \frac{2}{3}m & \frac{4}{3}m \\ 0 & -1 & \frac{1}{2}-m & 1-2m \\ 0 & -\frac{5}{2} & \frac{3}{2}-m & \frac{1}{2}-2m \end{array}\right) \begin{array}{l} \\ :(-1) \\ \\ \end{array} \end{array}$$

$$\to \begin{array}{c} \begin{array}{ccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 2 & \frac{2}{3}m & \frac{4}{3}m \\ 0 & 1 & m-\frac{1}{2} & 2m-1 \\ 0 & -\frac{5}{2} & \frac{3}{2}-m & \frac{1}{2}-2m \end{array}\right) \begin{array}{l} -2(II) \\ :(-1) \\ +\frac{5}{2}(II) \end{array} \end{array}$$

$$\to \begin{array}{c} \begin{array}{ccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1-\frac{4}{3}m & 2-\frac{8}{3}m \\ 0 & 1 & m-\frac{1}{2} & 2m-1 \\ 0 & 0 & \frac{1}{4}+\frac{3}{2}m & 3m-2 \end{array}\right) \end{array} .$$

To determine m such that the payoff

$$\begin{pmatrix} 2m \\ 1.0 \\ 0.5 \end{pmatrix}$$

can be achieved by linearly combining the three assets, we need to *exclude* the case where the system has no solutions. This is verified when $\mathrm{rg}(A) < \mathrm{rg}(A, \mathbf{b})$, i.e.,

$$\frac{1}{4} + \frac{3}{2}m = 0 \text{ and } 3m - 2 \neq 0 \Leftrightarrow m = -\frac{1}{6}.$$

It follows that the payoff

$$\begin{pmatrix} 2m \\ 1.0 \\ 0.5 \end{pmatrix}$$

can be achieved by linearly combining the three assets if and only if $m \neq -\frac{1}{6}$.

(c) (10 points)
In this exercise the key word to understand which mathematical framework could be applied to solve the problem is "maximize". The problem is an optimisation problem, and thus either an optimisation with constraints or without constraints. Constraints are conditions that limit the choice of the variables. However, in this case variables a and p can be freely chosen and thus we are facing an optimisation without constraints. Therefore, the next question is which function needs to be maximised. The text mentions that the profit as to be maximised and this corresponds to revenues minus costs. We have:

Revenues:

$$\underbrace{(3,000 + 4\sqrt{a} - 20\,p)}_{\text{demand}}\ \underbrace{p}_{\text{price}}$$

Costs:

$$\underbrace{20,000}_{\text{fixed costs}} + \underbrace{2\,(3,000 + 4\sqrt{a} - 20\,p)}_{\text{production costs}} + \underbrace{a}_{\text{advertising}}$$

It follows that:

$$\begin{aligned} f(a,p) &= \textbf{Profit} \\ &= \textbf{Revenues} - \textbf{Costs} \\ &= (3,000 + 4\sqrt{a} - 20\,p)\,p - \left[20,000 + 2\,(3,000 + 4\sqrt{a} - 20\,p) + a\right] \\ &= (3,000 + 4\sqrt{a} - 20\,p)\,(p-2) - 20,000 - a. \end{aligned}$$

The <u>necessary conditions</u> for maxima, minima or saddle points (a_0, p_0) of f are

$$\begin{cases} f_a(a_0,p_0) = 0 \\ f_p(a_0,p_0) = 0 \end{cases}.$$

We thus compute the first order partial derivatives of f and we obtain

$$\begin{aligned} f_a(a,p) &= \frac{4}{2\sqrt{a}}\,(p-2) - 1 = \frac{2}{\sqrt{a}}\,(p-2) - 1, \\ f_p(a,p) &= -20\,(p-2) + (3,000 + 4\sqrt{a} - 20\,p) = -40\,p + 3,040 + 4\sqrt{a}. \end{aligned}$$

It follows that:

$$\begin{cases} f_a(a_0,p_0) = 0 \\ f_y(a_0,p_0) = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{2}{\sqrt{a_0}}\,(p_0-2) - 1 = 0 \\ -40\,p_0 + 3,040 + 4\sqrt{a_0} = 0 \end{cases}.$$

From $\frac{2}{\sqrt{a_0}}\,(p_0-2) - 1 = 0$ we obtain

$$\sqrt{a_0} = 2\,(p_0 - 2).$$

We plug this latter result into $-40\,p_0 + 3,040 + 4\sqrt{a_0} = 0$ and we find:

$$\begin{aligned} -40\,p_0 + 3,040 + 4\sqrt{a_0} = 0 &\overset{\sqrt{a_0}=2\,(p_0-2)}{\Leftrightarrow} -40\,p_0 + 3,040 + 8\,(p_0-2) = 0 \\ &\Leftrightarrow 32\,p_0 + 3,024 = 0 \\ &\Leftrightarrow p_0 = 94.5. \end{aligned}$$

It follows that:

$$\sqrt{a_0} = 2\,(p_0-2) = 2 \cdot (94.5 - 2) = 185 \Leftrightarrow a_0 = 34,225.$$

Next, we verify the <u>sufficient conditions</u>: if (a_0, p_0) satisfies the necessary conditions, then the following holds:

$$\left\{\begin{array}{c} f_{aa}(a_0,p_0) > 0 \\ f_{pp}(a_0,p_0) > 0 \\ f_{aa}(a_0,p_0)\, f_{pp}(a_0,p_0) - (f_{ap}(a_0,p_0))^2 > 0 \end{array}\right. \qquad \Rightarrow (a_0,p_0) \text{ is a minimum,}$$

$$\left\{\begin{array}{c} f_{aa}(a_0,p_0) < 0 \\ f_{pp}(a_0,p_0) < 0 \\ f_{aa}(a_0,p_0)\, f_{pp}(a_0,p_0) - (f_{ap}(a_0,p_0))^2 > 0 \end{array}\right. \qquad \Rightarrow (a_0,p_0) \text{ is a maximum,}$$

and

$$f_{aa}(a_0,p_0)\, f_{pp}(a_0,p_0) - (f_{ap}(a_0,p_0))^2 < 0 \Rightarrow (a_0,p_0) \text{ is a saddle point.}$$

For the sufficient conditions we need the second order partial derivatives of f. We have:

$$\begin{aligned} f_{aa}(a,p) &= 2\left(-\frac{1}{2}\right) a^{-\frac{3}{2}}\,(p-2) = -a^{-\frac{3}{2}}\,(p-2), \\ f_{pp}(a,p) &= -40, \\ f_{ap}(a,p) &= \frac{2}{\sqrt{a}} = 2\,a^{-\frac{1}{2}}. \end{aligned}$$

It follows that $f_{aa}(a,p) < 0$ if $a > 0$ and $p > 2$, and $f_{pp}(a,p) < 0$ for all a and p. Moreover,

$$f_{aa}(a_0,p_0)\, f_{pp}(a_0,p_0) - (f_{ap}(a_0,p_0))^2 > 0.$$

for $(a_0,p_0) = (34,225,94.5)$.

Therefore, $(a_0,p_0) = (34,225,94.5)$ is a maximum of f.

(d) (7 points)

In this exercise the key word to understand which mathematical framework could be applied to solve the problem is "extreme (maximal or minimal)". The problem is an optimisation problem, and thus either an optimisation with constraints or without constraints. Constraints are conditions that limit the choice of the variables. In this case variables x and y *cannot* be freely chosen, because the distance between point (x,y) and point $(a,0)$ must corresponds to 4, i.e., point (x,y) must lie on the circle with center $(a,0)$ and radius 4 (a picture helps visualising the situation). Therefore, we are facing an optimisation with constraints. The next questions are which function needs to be maximised/minimised and what is the constraint. The text mentions that the area of R needs to be maximised/minimised and this corresponds to:

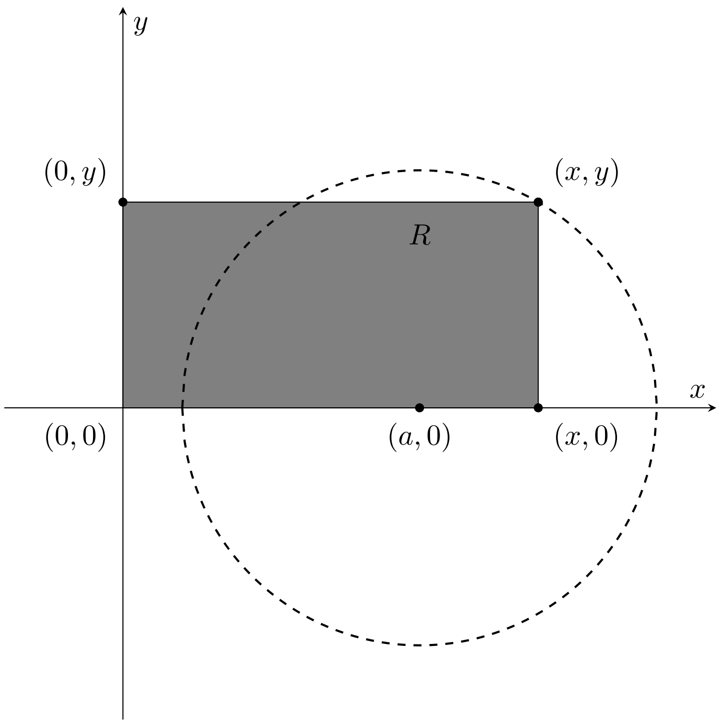
$$f(x,y) = x\,y.$$

Finally, the constraint can be specified as:

$$\sqrt{(x-a)^2+y^2} = 4 \Leftrightarrow (x-a)^2+y^2 = 4^2 \Leftrightarrow \varphi(x,y) = (x-a)^2+y^2-16 = 0.$$

Therefore, the optimisation problem can now be formalised as follows:

$$\max f(x,y) = x\,y \text{ such that } \varphi(x,y) = (x-a)^2+y^2-16 = 0.$$



To solve this problem we apply the Lagrange method. First of all we define the Lagrange function:

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda\, \varphi(x, y) \\ &= x\, y + \lambda\, ((x-a)^2 + y^2 - 16). \end{aligned}$$

The necessary conditions for constrained extreme points of f, given the constraint $\varphi(x, y) = 0$, are the so-called Lagrange conditions:

$$F_x(x, y, \lambda) = 0 \quad \Rightarrow \quad y + 2\,\lambda\,(x-a) = 0, \qquad \text{(I)}$$
$$F_y(x, y, \lambda) = 0 \quad \Rightarrow \quad x + 2\,\lambda\, y = 0, \qquad \text{(II)}$$
$$F_\lambda(x, y, \lambda) = 0 \quad \Rightarrow \quad (x-a)^2 + y^2 - 16 = 0. \qquad \text{(III)}$$

From (I) we obtain

$$2\,\lambda = -\frac{y}{x-a}. \qquad \text{(IV)}$$

From (II) we obtain

$$2\,\lambda = -\frac{x}{y}. \qquad \text{(V)}$$

Equations (IV) and (V) imply that:

$$-\frac{y}{x-a} = 2\,\lambda = -\frac{x}{y} \Leftrightarrow y^2 = x\,(x-a).$$

We plug this result into (III) and we have:

$$(x-a)^2 + x\,(x-a) - 16 = 0 \Leftrightarrow 2\,x^2 - 3\,a\,x + a^2 - 16 = 0 \Leftrightarrow x = \frac{3\,a \pm \sqrt{9\,a^2 - 8\,(a^2-16)}}{4} = \frac{3\,a \pm \sqrt{a^2+128}}{4}.$$

Because $x = \frac{3\,a - \sqrt{a^2+128}}{4} \in \left(0, \frac{a}{2}\right)$ for $a \in (4, 8)$, then $x\,(x-a) < 0$, which causes a contradiction to

$y^2 = x\,(x-a)$. Therefore, $x = \frac{3\,a-\sqrt{a^2+128}}{4}$ must be excluded and the only solution is

$$x = \frac{3\,a+\sqrt{a^2+128}}{4}.$$

It then follows that:

$$y = \sqrt{x\,(x-a)} = \sqrt{\frac{3\,a+\sqrt{a^2+128}}{4}\left(\frac{3\,a+\sqrt{a^2+128}}{4}-a\right)} = \sqrt{\frac{64+a\,\sqrt{a^2+128}-a^2}{8}}.$$

Part II: Multiple-choice Questions

Exercise 2

	(a)	(b)	(c)	(d)
Question 1	☐	☐	☐	☒
Question 2	☐	☐	☐	☒
Question 3	☐	☒	☐	☐
Question 4	☒	☐	☐	☐
Question 5	☐	☐	☐	☒
Question 6	☐	☐	☐	☒
Question 7	☐	☒	☐	☐
Question 8	☐	☒	☐	☐
Question 9	☐	☐	☒	☐
Question 10	☒	☐	☐	☐
Question 11	☐	☒	☐	☐
Question 12	☐	☐	☒	☐

Q1. (d). The constraint $\varphi(x,y) = \frac{x^2}{36} + \frac{(y-3)^2}{16} = 1$ describes an ellipse with center $(0,3)$ and semi-axis $a = 6$ and $b = 4$. Therefore, $P = (6,3)$ is the point on this ellipse with the highest possible value for first coordinate. It follows that $P = (6,3)$ is a maximum of $f(x,y) = x$ under the constraint $\varphi(x,y) = \frac{x^2}{36} + \frac{(y-3)^2}{16} = 1$.

Q2. (d). First of all, $(-x_0, -y_0)$ is a stationary point of g because (x_0, y_0) is a stationary point of f. Indeed,

$$g_x(-x_0,-y_0) = f_x(x_0,y_0) = 0$$

and

$$g_y(-x_0,-y_0) = f_y(x_0,y_0) = 0.$$

Moreover, $(-x_0,-y_0)$ is a local minimum of g because (x_0,y_0) is a local maximum of f. Indeed,

$$g_{xx}(-x_0,-y_0) = -f_{xx}(x_0,y_0) > 0,$$

$$g_{yy}(-x_0,-y_0) = -f_{yy}(x_0,y_0) > 0,$$

and

$$g_{xx}(-x_0,-y_0)\, g_{yy}(-x_0,-y_0) - (g_{xy}(-x_0,-y_0))^2 = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2 > 0.$$

Q3. (b). Because $(x_0,y_0) = (1,2)$ is a local maximum of f under the constrain $\varphi(x,y) = 0$, then the slope of the tangent line to the contour line of f at $(x_0,y_0) = (1,2)$ corresponds to the slope the tangent line to the contour line $\varphi(x,y) = 0$. Applying the implicit function theorem we obtain:

$$\frac{dy}{dx}(1,2) = -\frac{\varphi_x(1,2)}{\varphi_y(1,2)} = -\frac{2}{3}.$$

Q4. (a). The integral function I is an antiderivative of f, i.e., $I'(x) = f(x)$.

Q5. (d). The only correct answer is (d), which corresponds to the formula for integration by substitution. Note that $F'(x) = f(x)$, because F is an antiderivative of f. Differentiating each equation left and right of the equality sign we obtain:

(a) $f(g(x)) = f(x)$, which is generally wrong;

(b) $f(g(x)) = f(g(x))\, g'(x)$ which is generally wrong;
(c) $f(g(x))\, f'(x) = f(g(x))\, g'(x)$, which is generally wrong;
(d) $f(g(x))\, g'(x) = f(g(x))\, g'(x)$, which is clearly correct.

Q6. (d). We have:

(a) $(x^2 + x\, e^{x^2} + C)' = 2\, x + (1 + 2\, x^2)\, e^{x^2}$;
(b) $(3\, x + 2\, x\, e^{x^2} + C)' = 3 + (2 + 4\, x^2)\, e^{x^2}$;
(c) $(3\, x^2 + 2\, x\, e^{x^2} + C)' = 6\, x + (2 + 4\, x^2)\, e^{x^2}$;
(d) $(3\, x^2 + x\, e^{x^2} + C)' = 6\, x + (1 + 2\, x^2)\, e^{x^2}$.

Therefore, only the function in (d) is antiderivative of the integrand of the given integral. This implies:

$$\int \left[6\, x + (1 + 2\, x^2)\, e^{x^2}\right]\, dx = 3\, x^2 + x\, e^{x^2} + C.$$

Q7. (b). A density function is a non-negative function with $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Clearly, in our case, $f(x) \geq 0$ for all x when $a > 0$. Moreover,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 f(x)\, dx = \int_0^1 \left(a\, x^2 + \frac{1}{2}\right)\, dx = \frac{a}{3} + \frac{1}{2}.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \Leftrightarrow \frac{a}{3} + \frac{1}{2} = 1 \Leftrightarrow a = \frac{3}{2}.$$

Q8. (b). A density function is a non-negative function with $\int_{-\infty}^{\infty} f(x)\, dx = 1$. We have:

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^8 f(x)\, dx = \int_0^8 \left(a\, x + \frac{1}{16}\right)\, dx = 32\, a + \frac{1}{2}.$$

Therefore,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \Leftrightarrow 32\, a + \frac{1}{2} = 1 \Leftrightarrow a = \frac{1}{64}.$$

Clearly, for $a = \frac{1}{64}$ we also have $f(x) \geq 0$ for all $x \in \mathbb{R}$ and thus f is a density function. It follows that:

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x\, f(x)\, dx \\
&= \int_0^8 x \left(\frac{x}{64} + \frac{1}{16}\right)\, dx \\
&= \int_0^8 \left(\frac{x^2}{64} + \frac{x}{16}\right)\, dx \\
&= \left[\frac{x^3}{192} + \frac{x^2}{32}\right]_0^8 \\
&= \frac{512}{192} + \frac{64}{32} \\
&= \frac{896}{192} \\
&= \frac{14}{3}.
\end{aligned}$$

Q9. (c). We have:

$$\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + \underbrace{\mathbf{a}\cdot\mathbf{b}}_{=0 \text{ because are orthogonal}} = ||\mathbf{a}||^2 = 9.$$

Q10. (a). A basis of a vector space is a system of linearly independent vectors that spans the vector space. Therefore, we can exclude (d) because in $\mathbb{R}^3$ the maximum number of linearly independent vectors is 3. We now check (a), (b), and (c). We apply the following result: 3 vectors in $\mathbb{R}^3$ defines a basis of $\mathbb{R}^3$ if and only if they are linearly independent, i.e., the 3×3 matrix with the columns given by the three vectors is regular, or equivalently its determinant is different from zero. We have:

(a) $\left|\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix}\right| = -2 \neq 0.$

(b) $\left|\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 4 \\ -1 & 3 & 8 \end{pmatrix}\right| = 0.$

(c) $\left|\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix}\right| = 0.$

Q11. (b). Existence of infinitely many solutions implies $\mathrm{rg}(A) = \mathrm{rg}(A,\mathbf{b}) < n = 5$. In this case, the dimension of the solution space is $3 = 5 - \mathrm{rg}(A)$. Therefore, $\mathrm{rg}(A) = 2$.

Q12. (c). We have:

$$\begin{aligned}
B^T\,(A\,B)^T\,(B^{-1}\,A^{-1})^T\,B\,(A\,B)^{-1} &= B^T\,B^T\,A^T\,(A^{-1})^T\,(B^{-1})^T\,\underbrace{B\,B^{-1}}_{=I}\,A^{-1} \\
&= B^T\,B^T\,\underbrace{A^T\,(A^T)^{-1}}_{=I}\,(B^T)^{-1}\,A^{-1} \\
&= B^T\,\underbrace{B^T\,(B^T)^{-1}}_{=I}\,A^{-1} \\
&= B^T\,A^{-1} \\
&\overset{A \text{ symmetric}}{=} B^T\,(A^{-1})^T \\
&= (A^{-1}\,B)^T.
\end{aligned}$$

Exercise 3

	(a)	(b)	(c)	(d)
Question 1	☐	☒	☐	☐
Question 2	☐	☐	☒	☐
Question 3	☐	☒	☐	☐
Question 4	☐	☒	☐	☐
Question 5	☐	☒	☐	☐
Question 6	☐	☐	☒	☐
Question 7	☐	☐	☐	☒
Question 8	☐	☐	☒	☐

Q1. (b). Because f is the density function of random variable X on $[0,1]$ we have:

$$\int_0^1 f(x)\,dx = 1$$

and

$$\int_0^1 x\,f(x)\,dx = \mathbb{E}\big[X\big].$$

It follows that:

$$\int_0^1 (x+1)\,f(x)\,dx = \underbrace{\int_0^1 x\,f(x)\,dx}_{=\mathbb{E}[X]} + \underbrace{\int_0^1 f(x)\,dx}_{=1} = \frac{28}{45} + 1 = \frac{73}{45}.$$

Q2. (c). $\mathbf{u}$ is orthogonal to $\mathbf{v}$ if and only if

$$\begin{aligned} &\mathbf{u}\cdot\mathbf{v} = 0 \\ &\quad\Leftrightarrow\quad t\,(t+5) + (t-1)\,(-1) + (-6)\cdot 1 = 0 \\ &\quad\Leftrightarrow\quad t^2 + 4\,t - 5 = 0 \\ &\quad\Leftrightarrow\quad (t+5)\,(t-1) = 0 \\ &\quad\Leftrightarrow\quad t \in \{-5, 1\}. \end{aligned}$$

For $t=-5$, we have

$$\mathbf{u} = \begin{pmatrix} -5 \\ -6 \\ -6 \end{pmatrix}$$

and

$$||\mathbf{u}|| = \sqrt{25+36+36} = \sqrt{97}.$$

For $t=1$, we have

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$$

and

$$||\mathbf{u}|| = \sqrt{1+0+36} = \sqrt{37}.$$

Q3. (b). We apply the Gauss method:

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix} \begin{matrix} \\ -9\,(I) \\ -7\,(I) \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -13 & -7 & 11 \\ 0 & -13 & -7 & 11 \end{pmatrix} \begin{matrix} \\ \\ -(II) \end{matrix}$$

$$A^\star = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 0 & -13 & -7 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

It follows that $\text{rg}(A^\star) = 2$ and thus $\text{rg}(A) = \text{rg}(A^\star) = 2$.

Q4. (b). To determine the inverse of A we apply Gauss:

$$(A, I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ \\ -(I) \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} \\ :(-1) \\ \\ \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} \\ \\ :(-1) \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \begin{matrix} -(III) \\ \\ \\ \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) = (I, A^{-1}).$$

It follows that:

$$A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} .$$

Q5. (b). If λ is an eigenvalue of A associated with eigenvector $\mathbf{x}$ then λ^2 is an eigenvalue of A^2 associated with eigenvector $\mathbf{x}$. This can be easily seen as follows:

$$A^2\,\mathbf{x} = A\,(A\,\mathbf{x}) = A(\lambda\,\mathbf{x}) = \lambda\,A\,\mathbf{x} = \lambda\,(\lambda\,\mathbf{x}) = \lambda^2\,\mathbf{x}.$$

We compute the eigenvalues of A. We have:

$$\det(A-\lambda I) = (1-\lambda)^3 - (1-\lambda) = (1-\lambda)\left((1-\lambda)^2-1\right) = \lambda\,(1-\lambda)(\lambda-2).$$

Therefore,

$$\det(A-\lambda I) = 0 \Leftrightarrow \lambda \in \{0,1,2\},$$

i.e., $0,1,2$ are the eigenvalues of A. If follows that 0,1,4 are the eigenvalues of A^2.

Q6. (c). Answers (a), (b), and (d) do not satisfy the condition $y_0 = 2$. For answer (c) we have:

$$y_{k+1}-(1+a)\,y_k = 4\,(1+a)^{k+1}-2-(1+a)\,(4\,(1+a)^k-2) = 4\,(1+a)^{k+1}-2-4\,(1+a)^{k+1}+2+2\,a = 2\,a$$

for $a \neq 1$ and $a \neq 0$. Therefore, (c) also satisfy the difference equation.

Q7. (d). The normal form of the difference equation is

$$y_{k+1} = 2\,y_k + 7.$$

It follows that $A = 2$ and $B = 7$. Because $A > 0$ and $|A| > 1$, then the general solution of the difference equation is monotone and divergent.

Q8. (b). The normal form of the difference equation is

$$y_{k+1} = \frac{1}{a+2}\,y_k - \frac{a^2-4}{a+2},$$

i.e., $A = \frac{1}{a+2}$ and $B = -\frac{a^2-4}{a+2} = 2-a$. Because $A \neq 1$, then the general solution of the difference equation is monotone and convergent if and only if $0 < A < 1$. We have:

$$0 < A < 1 \Leftrightarrow (a+2) > 0 \text{ and } a+2 > 1 \Leftrightarrow a > -1.$$

It follows that the general solution of the difference equation is monotone and convergent if and only if $a > -1$. Furthermore, the solution converges to 0 if and only if

$$\lim_{k\to\infty} y_k = y^\star = \frac{B}{1-A} = \frac{2-a}{1-\frac{1}{a+2}} = -\frac{a^2-4}{a+1} = 0 \overset{a>-1}{\Longleftrightarrow} a = 2.$$