Mathematics B
Master Solution Alternative Date Exam
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Part I: Open Questions

Open questions require more than just applying a set of formulas or equations. Therefore, simply searching for the right formula is very likely not a successful strategy. By contrast, when reading the problem one should first identify key words that point to the most useful mathematical framework, and then, in a second stage, verify if the information provided allows to apply specific mathematical results. For example, if the text refers to "maximization" or "minimization", then very likely one has to apply mathematical optimisation and thus verify if this refers to constrained or to unconstrained optimisation. Translating the verbal description of a problem into an equivalent mathematical formulation of it is called *mathematical formalisation* and is a crucial step to make use of mathematics for solving problems. Open questions require that problems are formalised using mathematics before the adequate mathematical results and equations can be applied.

Exercise 1

(a) (8 points)

In this exercise the key words to understand which mathematical framework could be applied to solve the problem are "value of the deposit at time T" and "continuously compounded interest rate". The final value corresponds to the compounded sum of cash flows or, equivalently, to the present value of all cash flows compounded to $T = 20$ (Section 8.2). However, because cash flows are generated continuously, here we need to replace the sum with an integral and use the exponential function to compound cash flows to the time point $T = 20$ (section 15.2). This leads us to the following mathematical equation for the final value:

$$FV(20) = e^{i\,T}\,PV(20) = e^{i\,T}\int_0^T D(t)\,e^{-i\,t}\,dt = e^{20\,i}\int_0^{20}(b\,t+200)\,e^{-i\,t}\,dt.$$

Before solving the integral, we need to understand what are we looking for. Using the above formalisation, we see that the problem is about determining the parameter b such that

$$FV(20) = 100,000$$

or equivalently,

$$PV(20) = 100,000\,e^{-20\,i}.$$

At this stage the plan is clear: (i) solve the integral to obtain an explicit formula for $PV(20)$, (ii) use the condition $PV(20) = 100,000\,e^{-20\,i}$ to determine the parameter b.
To solve the integral $\int_0^{20}(b\,t+200)\,e^{-i\,t}\,dt$ we apply integration by parts with $u(t) = b\,t+200$ and $v'(t) = e^{-i\,t}$. It follows that $u'(t) = b$ and $v(t) = -\frac{1}{i}\,e^{-i\,t}$. Therefore:

$$\begin{aligned}
\int_0^{20}\underbrace{(b\,t+200)}_{=u(t)}\underbrace{e^{-i\,t}}_{=v'(t)}\,dt &= \underbrace{(b\,t+200)}_{=u(t)}\underbrace{\left(-\frac{1}{i}\,e^{-i\,t}\right)}_{=v(t)}\Bigg|_0^{20} - \int_0^{20}\underbrace{b}_{=u'(t)}\underbrace{\left(-\frac{1}{i}\,e^{-i\,t}\right)}_{=v(t)}\,dt \\
&\overset{i=0.05}{=} -20\,e^{-1}(20\,b+200) + 20\cdot 200 + 20\,b\int_0^{20} e^{-0.05\,t}\,dt \\
&= -400\,e^{-1}\,b - 4000\,e^{-1} + 4000 + 20\,b\,(-20)\,e^{-0.05\,t}\Big|_0^{20} \\
&= -400\,e^{-1}\,b + 4000\,(1-e^{-1}) - 400\,e^{-1}\,b + 400\,b \\
&= b\,(400 - 800\,e^{-1}) + 4000\,(1-e^{-1}).
\end{aligned}$$

It follows that:

$$\begin{aligned} 100,000\, e^{-20\cdot 0.05} &= b\,(400 - 800\, e^{-1}) + 4000\,(1 - e^{-1}) \\ \Rightarrow b &= \frac{100,000 - 4000\,(e-1)}{400\, e - 800} \approx 324.13. \end{aligned}$$

(b) (8 points)

In this exercise the key words to understand which mathematical framework could be applied to solve the problem are "table", "generate the same payoff" and "Gaussian elimination". In mathematics, tables of numbers corresponds to matrices (Section 16.1) and "generate the same payoff" refers to the question of whether a system of linear equations possess solutions (Chapter 18).

Specifically, the payoff

$$\begin{pmatrix} 1 \\ 0 \\ m+2 \\ 1 \end{pmatrix}$$

can be achieved by linearly combining the four assets if and only if the system

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 \\ 1 \\ m^2 - m - 11 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ m+2 \\ 1 \end{pmatrix}$$

possesses at least one solution $(\lambda_1^\star, \lambda_2^\star, \lambda_3^\star, \lambda_4^\star)$. Because systems of linear equations possess either a unique solution, or infinite many solutions, or no solution, then we need to identify the values for m for which the system has either a unique solution or infinite many solutions. To achieve this we apply Gauss:

$$(A, \mathbf{b}) = \left(\begin{array}{cccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & m^2 - m - 11 & m+2 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} \\ \\ -2\,(I) \\ -(I) \\ -(I) \end{array}$$

$$\rightarrow \left(\begin{array}{cccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \\ 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & m^2 - m - 11 & m+1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \begin{array}{c} \\ \\ \\ -(II) \\ \\ \end{array}$$

$$\rightarrow \left(\begin{array}{cccc|c} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \\ 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & m^2 - m - 12 & m+2 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

To determine m such that the payoff

$$\begin{pmatrix} 1 \\ 0 \\ m+2 \\ 1 \end{pmatrix}$$

can be achieved by linearly combining the four assets, we need to *exclude* the case where the system has no solutions. This is verified when $\mathrm{rg}(A) < \mathrm{rg}(A, \mathbf{b})$, i.e.,

$$m^2 - m - 12 = (m+3)\,(m-4) = 0 \;\text{ and }\; m+2 \neq 0 \;\Leftrightarrow\; m = -3 \;\text{ or }\; m = 4.$$

It follows that the payoff

$$\begin{pmatrix} 1 \\ 0 \\ m+2 \\ 1 \end{pmatrix}$$

can be achieved by linearly combining the four assets if and only if $m \notin \{-3, 4\}$.

(c) (10 points)

In this exercise the key word to understand which mathematical framework could be applied to solve the problem is "maximize". The problem is an optimisation problem, and thus either an optimisation with constraints or without constraints. Constraints are conditions that limit the choice of the variables. However, in this case variables I and p can be freely chosen and thus we are facing an optimisation without constraints. Therefore, the next question is which function needs to be maximised. The text mentions that the profit has to be maximised and this corresponds to revenues minus costs. We have:

Revenues:

$$\underbrace{(1000 + 5\sqrt{I} - 40\,p)}_{\text{demand}}\;\underbrace{p}_{\text{price}}$$

Costs:

$$\underbrace{I}_{\text{initial investment}} \quad + \underbrace{0.2 \cdot (1000 + 5\sqrt{I} - 40\,p)\,p}_{\text{maintenance costs}}$$

It follows that:

$$\begin{aligned} f(I,p) &= \textbf{Profit} \\ &= \textbf{Revenues} - \textbf{Costs} \\ &= (1000 + 5\sqrt{I} - 40\,p)\,p - \Big[I + 0.2 \cdot (1000 + 5\sqrt{I} - 40\,p)\,p\Big] \\ &= 0.8 \cdot (1000 + 5\sqrt{I} - 40\,p)\,p - I. \end{aligned}$$

The <u>necessary conditions</u> for maxima, minima or saddle points (I_0, p_0) of f are

$$\begin{cases} f_I(I_0, p_0) = 0 \\ f_p(I_0, p_0) = 0 \end{cases}.$$

We thus compute the first order partial derivatives of f and we obtain

$$\begin{aligned} f_I(I,p) &= 0.8\,\frac{5}{2}\,I^{-\frac{1}{2}}\,p - 1, \\ f_p(I,p) &= 0.8\,(-40)\,p + 0.8\,(1000 + 5\sqrt{I} - 40\,p). \end{aligned}$$

It follows that:

$$\begin{cases} f_I(I_0,p_0)=0 \\ f_p(I_0,p_0)=0 \end{cases} \Leftrightarrow \begin{cases} 2\,I_0^{-\frac{1}{2}}\,p_0-1=0 \\ -32\,p_0+800+4\sqrt{I_0}-32\,p_0=0 \end{cases}.$$

From $2\,I_0^{-\frac{1}{2}}\,p_0-1=0$ we obtain

$$\sqrt{I_0}=2\,p_0.$$

We plug this latter result into $-32\,p_0+800+4\sqrt{I_0}-32\,p_0=0$ and we find:

$$\begin{aligned} -32\,p_0+800+4\sqrt{I_0}-32\,p_0=0 &\overset{\sqrt{I_0}=2\,p_0}{\Leftrightarrow} \quad -32\,p_0+800+8\,p_0-32\,p_0=0 \\ &\Leftrightarrow \quad p_0=\frac{800}{56}=\frac{100}{7} \end{aligned}$$

It follows that:

$$\sqrt{I_0}=2\cdot\frac{100}{7}\Rightarrow I_0=\left(\frac{200}{7}\right)^2.$$

Next, we verify the <u>sufficient conditions</u>: if (I_0,p_0) satisfies the necessary conditions, then the following holds:

$$\begin{cases} f_{II}(I_0,p_0)>0 \\ f_{pp}(I_0,p_0)>0 \\ f_{II}(I_0,p_0)\,f_{pp}(I_0,p_0)-(f_{Ip}(I_0,p_0))^2>0 \end{cases} \Rightarrow (I_0,p_0) \text{ is a minimum,}$$

$$\begin{cases} f_{II}(I_0,p_0)<0 \\ f_{pp}(I_0,p_0)<0 \\ f_{II}(I_0,p_0)\,f_{pp}(I_0,p_0)-(f_{Ip}(I_0,p_0))^2>0 \end{cases} \Rightarrow (I_0,p_0) \text{ is a maximum,}$$

and

$$f_{II}(I_0,p_0)\,f_{pp}(I_0,p_0)-(f_{Ip}(I_0,p_0))^2<0\Rightarrow (I_0,p_0) \text{ is a saddle point.}$$

For the sufficient conditions we need the second order partial derivatives of f. We have:

$$\begin{aligned} f_{II}(I,p) &= (-1)\,I^{-\frac{3}{2}}\,p<0, \\ f_{pp}(I,p) &= -32-32=-64<0, \\ f_{Ip}(I,p) &= 2\,I^{-\frac{1}{2}}. \end{aligned}$$

Plugging (I_0,p_0) into the second order condition, we get

$$\Rightarrow f_{II}\left(\left(\frac{200}{7}\right)^2,\frac{100}{7}\right)\cdot f_{pp}\left(\left(\frac{200}{7}\right)^2,\frac{100}{7}\right)-\left(f_{pI}\left(\left(\frac{200}{7}\right)^2,\frac{100}{7}\right)\right)^2$$

$$=(-1)\left(\frac{200}{7}\right)^{-3}\cdot\frac{100}{7}\cdot(-64)-\left(2\cdot\left(\frac{200}{7}\right)^{-1}\right)^2=\frac{49}{2500}>0.$$

Therefore, $(I_0,p_0)=\left(\left(\frac{200}{7}\right)^2,\frac{100}{7}\right)$ is a maximum of f.

(d) (7 points)

In this exercise the key word to understand which mathematical framework could be applied to solve the problem is “minimize”. The problem is an optimisation problem, and thus either an optimisation with constraints or without constraints. Constraints are conditions that limit the choice of the variables. In this case variables h and r *cannot* be freely chosen, because the capacity (volume) of each container is a function of its height and radius and must correspond to $128\,cm^3$. Therefore, we are facing an optimisation with constraints. The next questions are which function needs to be maximised/minimised and what is the constraint. The text mentions that the surface of the containers needs to be minimised and it corresponds to:

$$f(r,h) = \underbrace{2\,r^2\,\pi}_{\text{bottom and top base}} + \underbrace{2\,r\,\pi\,h}_{\text{lateral surface}}.$$

Finally, the constraint can be specified as:

$$r^2\,\pi\,h = 128 \;\Leftrightarrow\; \varphi(x,y) = r^2\,\pi\,h - 128 = 0.$$

Therefore, the optimisation problem can now be formalised as follows:

$$\max f(r,h) = 2\,r^2\,\pi + 2\,r\pi\,h \;\text{ such that }\; \varphi(r,h) = r^2\,\pi\,h - 128 = 0.$$

To solve this problem we apply the substitution method. Solving the constraint for h, we get

$$h = \frac{128}{r^2\,\pi} \qquad (*).$$

Plugging this into the objective function f we have:

$$f\left(r, \frac{128}{r^2\pi}\right) = F(r) = 2\,r^2\,\pi + 2\,r\,\pi\,\frac{128}{r^2\,\pi}.$$

The first order condition for a maximum of F corresponds to

$$F'(r) = 4\,r\,\pi + (-1)\cdot 256\,r^{-2} \stackrel{!}{=} 0,$$

from which we get

$$\frac{64}{\pi} = r^3 \;\Rightarrow\; r^\star = \sqrt[3]{\frac{64}{\pi}} = \frac{4}{\sqrt[3]{\pi}}.$$

Using $(*)$, the corresponding opimal hight is given as

$$h^\star = \frac{128}{(r^\star)^2\,\pi} = \frac{128\,\sqrt[3]{\pi^2}}{16\,\pi} = \frac{8}{\sqrt[3]{\pi}}.$$

(`Not requested:` Checking the second order condition

$$F''(r^\star) = 4\,\pi + 512\,(r^\star)^{-3} = 12\,\pi > 0$$

confirms that $(r^\star, h^\star)$ is indeed a minimum.)

We could also solve the problem with the Lagrange method. Therefore we define the Lagrange function:

$$\begin{aligned} F(r,h,\lambda) &= f(r,h) + \lambda\,\varphi(r,h) \\ &= 2\,r^2\,\pi + 2\,r\,\pi\,h + \lambda\,(r^2\,\pi\,h - 128). \end{aligned}$$

The necessary conditions for constrained extreme points of f, given the constraint $\varphi(r,h) = 0$, are the so-called Lagrange conditions:

$$F_r(r,h,\lambda) = 0 \;\Rightarrow\; 4\,r\,\pi + 2\,\pi\,h + 2\,\lambda\,r\,\pi\,h = 0, \tag{I}$$

$$F_h(r,h,\lambda) = 0 \;\Rightarrow\; 2\,r\,\pi + \lambda\,r^2\,\pi = 0, \tag{II}$$

$$F_\lambda(r,h,\lambda) = 0 \;\Rightarrow\; r^2\,\pi\,h - 128 = 0. \tag{III}$$

From (II) and $r > 0$ we obtain

$$r = -\frac{2}{\lambda}. \tag{IV}$$

From (III) we obtain

$$h = \frac{128}{r^2\,\pi}. \tag{V}$$

Using (IV) and (V) in (I) gives:

$$-\frac{8\,\pi}{\lambda} + \frac{256\,\lambda^2}{4} - \frac{256\,\lambda^2}{2} = 0 \;\Rightarrow\; 8\,\pi = -64\,\lambda^3.$$

It then follows that:

$$\begin{aligned} \lambda^\star &= -\frac{1}{2}\sqrt[3]{\pi}, \\ r^\star &= \frac{4}{\sqrt[3]{\pi}}, \\ h^\star &= \frac{8}{\sqrt[3]{\pi}}. \end{aligned}$$

Part II: Multiple-choice Questions

Exercise 2

	(a)	(b)	(c)	(d)
Question 1	☐	☐	☒	☐
Question 2	☐	☒	☐	☐
Question 3	☐	☐	☐	☒
Question 4	☐	☒	☐	☐
Question 5	☒	☐	☐	☐
Question 6	☐	☐	☒	☐
Question 7	☐	☒	☐	☐
Question 8	☐	☐	☒	☐
Question 9	☐	☐	☒	☐
Question 10	☒	☐	☐	☐
Question 11	☐	☒	☐	☐
Question 12	☐	☐	☒	☐

Q1. (c). The constraint $\varphi(x,y) = \frac{(x-2)^2}{36} + \frac{(y-4)^2}{9} = 1$ describes an ellipse with center $(2,4)$ and semi-axis $a = 6$ and $b = 3$. Therefore, $P = (2,7)$ is the point on this ellipse with the highest possible value for second coordinate. It follows that $P = (2,7)$ is a maximum of $f(x,y) = y$ under the constraint $\varphi(x,y) = \frac{(x-2)^2}{36} + \frac{(y-4)^2}{9} = 1$.

Q2. (b). First of all, $(-x_0, y_0)$ is a stationary point of g because (x_0, y_0) is a stationary point of f. Indeed,

$$g_x(-x_0, y_0) = -f_x(x_0, y_0)\,(-1) = 0$$

and

$$g_y(-x_0, y_0) = -f_y(x_0, y_0) = 0.$$

Moreover, $(-x_0, y_0)$ is a local minimum of g because (x_0, y_0) is a local maximum of f. Indeed,

$$g_{xx}(-x_0, y_0) = -f_{xx}(x_0, y_0) > 0,$$

$$g_{yy}(-x_0, y_0) = -f_{yy}(x_0, y_0) > 0,$$

and

$$g_{xx}(-x_0, y_0)\, g_{yy}(-x_0, y_0) - (g_{xy}(-x_0, y_0))^2 = f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0.$$

Q3. (d). Applying the implicit function theorem we obtain:

$$\begin{aligned}
\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} &= -\frac{\varphi_x(2,1)}{\varphi_y(2,1)} \\
&= -\left.\frac{2\,x + 3\ln y}{3\frac{x}{y} - 7}\right|_{(x,y)=(2,1)} \\
&= -\frac{4}{-1} \\
&= 4.
\end{aligned}$$

Q4. (b). The area of the region defined by the set

$$\{(x,y)\in\mathbb{R}^2 \,:\, a\le x\le b \text{ and } (f(x)\le y\le g(x) \text{ or } g(x)\le y\le f(x))\}$$

corresponds to the area between the graphs of the functions f and g between $x=a$ and $x=b$ and can be expressed by the integral

$$\int_a^b |f(x)-g(x)|\,dx.$$

Q5. (a). Using integration by parts with

$$\begin{aligned} u(x) &= \frac{1}{g(x)}, \quad \text{(chain rule)} \;\; u'(x) = -\frac{1}{[g(x)]^2}\cdot g'(x) \\ &\text{and} \\ v'(x) &= f'(x), \; v(x) = f(x), \end{aligned}$$

we get the expression in (a).

Q6. (d). We have:

$$\begin{aligned} (3x^2 + xe^{x^3} + C)' &= 6x + e^{x^3} + xe^{x^3}3x^2 \\ &= 6x + (3x^3+1)e^{x^3}. \end{aligned}$$

Q7. (b). A density function is a non-negative function with $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Clearly, in our case, $f(x)\ge 0$ for all x when $a>0$. Moreover, applying integration by parts gives

$$\begin{aligned} \int_1^{\sqrt{e}} ax\ln(x)\,dx &= a\left[\frac{1}{2}x^2\ln(x)\Big|_1^{\sqrt{e}} - \int_1^{\sqrt{e}} \frac{1}{2}x^2\frac{1}{x}\,dx\right] \\ &= a\left[\frac{1}{2}(\sqrt{e})^2\ln(\sqrt{e}) - 0 - \frac{1}{4}x^2\Big|_1^{\sqrt{e}}\right] \\ &= a\left[\frac{1}{4}e - \frac{1}{4}e + \frac{1}{4}\right] \\ &= a\frac{1}{4} \\ &\overset{!}{=} 1. \end{aligned}$$

Therefore, $a=4$.

Q9. (c).

$$\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot\mathbf{a} + \underbrace{\mathbf{a}\cdot\mathbf{b}}_{=0 \text{ (orthogonal)}} = ||\mathbf{a}||^2 = 16.$$

$$||\mathbf{a}|| = 4.$$

Q10. (a). A basis of a vector space is a system of linearly independent vectors that spans the vector space. Therefore, we can exclude (d) because in $\mathbb{R}^3$ the maximum number of linearly independent vectors is 3. We now check (a), (b), and (c). We apply the following result: 3 vectors in $\mathbb{R}^3$ defines a basis of $\mathbb{R}^3$ if and only if they are linearly independent, i.e., the 3×3 matrix with the columns given by the three vectors is regular, or equivalently its determinant is different from zero. We have:

(a) $\left|\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}\right| = 2 - 3 - 1 = -2 \neq 0.$

(b) $\left|\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 8 \end{pmatrix}\right| = 0.$ We have $\mathbf{a} + 3\,\mathbf{b} = \mathbf{d}$.

(c) $\left|\begin{pmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}\right| = 0.$ We have $\mathbf{b} - \mathbf{c} = \mathbf{e}$.

Q11. (b). The dimension of the solution space is equal to (a) the number of free variables describing the solution space and (b) the number of variables (columns of A) minus the rank of A. In this case we get $2 = 5 - \mathrm{rg}(A)$. Therefore, $\mathrm{rg}(A) = 3$.

Q12. (c). We use the following results for regular matrices S and R:

$$\begin{aligned} \det(S\,R) &= \det(S)\,\det(R), \\ \det(S^T) &= \det(S), \\ \det(S^{-1}) &= \frac{1}{\det(S)}. \end{aligned}$$

Hence it follows:

$$\begin{aligned} \det(B^T\,(AB)^T\,(B^{-1}A^{-1})^T\,B\,(AB)^{-1}) &= \det(B)\,\det(A)\,\det(B)\,\frac{1}{\det(B)}\,\frac{1}{\det(A)}\,\det(B)\,\frac{1}{\det(A)\det(B)} \\ &= \frac{\det(B)}{\det(A)}. \end{aligned}$$

Exercise 3

	(a)	(b)	(c)	(d)
Question 1	☐	☒	☐	☐
Question 2	☐	☐	☐	☒
Question 3	☐	☐	☒	☐
Question 4	☐	☐	☐	☒
Question 5	☐	☐	☒	☐
Question 6	☐	☐	☒	☐
Question 7	☐	☐	☐	☒
Question 8	☐	☐	☒	☐

Q1. (b). Because f is the density function of random variable X on $[0,1]$ we have:

$$\int_0^1 f(x)\,dx = 1$$

and

$$\int_0^1 x\,f(x)\,dx = \mathbb{E}\big[X\big].$$

It follows that:

$$5 = \int_0^1 (3\,x+2)\,f(x)\,dx = 3\underbrace{\int_0^1 x\,f(x)\,dx}_{=\mathbb{E}[X]} + 2\underbrace{\int_0^1 f(x)\,dx}_{=1}.$$

Hence, $\mathbb{E}\big[X\big] = \frac{5-2}{3} = 1$.

Q2. (c). $\mathbf{u}$ is orthogonal to $\mathbf{u}+\mathbf{v}$ if and only if

$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{u}+\mathbf{v}) &= \begin{pmatrix} t \\ t-1 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 2\,t+5 \\ t-2 \\ -5 \end{pmatrix} \\
&= 2\,t^2 + 5\,t + (t-1)(t-2) + 30 \\
&= 3\,t^2 + 2\,t + 32 \\
&\overset{!}{=} 0
\end{aligned}$$

Since the latter equation has no solution in the real numbers, there is no $t \in \mathbb{R}$ for which the vectors $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ are orthorgonal.

Q3. (c). We apply the Gauss method:

$$A = \begin{pmatrix} 1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 4 \end{pmatrix} \begin{matrix} \\ \\ -2\,(I) \end{matrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & -3 & -1 & 8 \end{pmatrix} \begin{matrix} -2\,(II) \\ \\ +3\,(II) \end{matrix}$$

$$A^\star = \begin{pmatrix} 1 & 0 & -3 & -8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 17 \end{pmatrix}.$$

It follows that $\text{rg}(A^\star) = 3$ (because it has a regular 3×3 submatrix) and thus $\text{rg}(A) = \text{rg}(A^\star) = 3$.

Q4. (d). The matrix A is regular (singular) if and only if $\det(A) \neq 0$ ($\det(A) = 0$). Using Sarrus' rule, we compute

$$\det(A) = m + 6 + 3\,m - m^2 - 3 - 6 = (-1)\,(m-3)\,(m-1).$$

Hence, $\det(A) = 0$ if and only if $m \in \{1, 3\}$.

Q5. (b). If λ is an eigenvalue of A associated with eigenvector $\mathbf{x}$ then λ^2 is an eigenvalue of A^2 associated with eigenvector $\mathbf{x}$. This can be easily seen as follows:

$$A^2\,\mathbf{x} = A\,(A\,\mathbf{x}) = A(\lambda\,\mathbf{x}) = \lambda\,A\,\mathbf{x} = \lambda\,(\lambda\,\mathbf{x}) = \lambda^2\,\mathbf{x}.$$

Using this result we get

$$\begin{aligned} A^2\mathbf{x} + A\mathbf{x} - 6\mathbf{x} = \mathbf{0} \Rightarrow \;\; \lambda^2\mathbf{x} + \lambda\mathbf{x} - 6\mathbf{x} &= \mathbf{0} \\ \Rightarrow \mathbf{x}(\lambda^2 + \lambda - 6) \quad = \mathbf{0} \Rightarrow \mathbf{x}(\lambda + 3)(\lambda - 2) &= \mathbf{0} \end{aligned}$$

$$\overset{\mathbf{x}\neq\mathbf{0}}{\Longrightarrow} \lambda_1 = -3, \quad \lambda_2 = 2$$

Comparing with the answers provided (and because λ_1 is not an eigenvalue of A), we get $\lambda = 2$.

Q6. (c). Answers (a), (b) do not satisfy the condition $y_0 = 6$. For answer (c) we have:

$$\begin{aligned} \sqrt{y_{k+1} - (1+a^2)\,y_k} &= \sqrt{10\,(1+a^2)^{k+1} - 4 - (1+a^2)\,(10\,(1+a^2)^k - 4)} \\ &= \sqrt{-4 - (1+a^2)\,(-4)} \\ &= 2\,a \end{aligned}$$

for $a > 0$. Therefore, (c) satisfies the difference equation. The same computations show that (d) violates the difference equation.

Q7. (d). We find the normal form of the difference equation:

$$2\,y_{k+1} = 9\,y_k + 3 \;\Rightarrow\; y_{k+1} = \frac{9}{2}\,y_k + \frac{3}{2}.$$

Since $\frac{9}{2} > 1$, it follows that the general solution is monotone and divergent.

Q8. (b). The normal form of the difference equation is

$$y_{k+1} = \frac{1}{a+3}\,y_k + 2 - a,$$

i.e., $A = \frac{1}{a+3}$ and $B = 2 - a$. Because $A \neq 1$, then the general solution of the difference equation is monotone and convergent if and only if $0 < A < 1$. We have:

$$0 < A < 1 \Leftrightarrow a + 3 > 0 \text{ and } a + 3 > 1 \Leftrightarrow a > -2.$$

It follows that the general solution of the difference equation is monotone and convergent if and only if $a > -2$. Furthermore, the solution converges to 1 if and only if

$$\lim_{k\to\infty} y_k = y^\star = \frac{B}{1-A} = \frac{2-a}{1 - \frac{1}{a+3}} = \frac{(2-a)\,(a+3)}{a+2} = 1.$$

Solving for a we get

$$a^2 + 2\,a - 4 = 0 \overset{a>-2}{\Longleftrightarrow} a = -1 + \sqrt{5}.$$