Mathematics B
Exam Spring Semester 2018

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Part I: Open Questions (36 points)

General instructions for the open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain a single version.

(vi) Temporary computations or sketches must be reported in separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (36 points)

(a) (8 points)

An investment generates the continuous cash flow $B(t) = a\,t + 10$ at time t for $t \in [0, 10]$, where $a > 0$. The continuously compounded interest rate corresponds to $i = 5\%$. Determine the parameter a such the present value of the investment corresponds to $1,000$ CHF.

Exercise 1

(a) (additional space for your solution)

Exercise 1

(b) (8 points)

The following table reports annual payoffs of three assets with same initial price depending on three different economic scenarios:

scenario	asset 1	asset 2	asset 3
expansion	1.5	3	m
stability	1.5	2	0.5
recession	1.5	0.5	1.5

An investor wants to linearly combine the three assets to obtain the following payoff:

scenario	investor's payoff
expansion	$2\,m$
stability	1.0
recession	0.5

Apply Gaussian elimination to determine the parameter values m such that investor's payoff can be achieved by combining the three existing assets.

Exercise 1

(b) (additional space for your solution)

Exercise 1 (36 points)

(c) (10 points)

The fixed costs to develop a new product are $20,000$ (CHF). Each unit of the product can be sold for p (CHF), while the production cost is 2 (CHF) per unit. You decide to start a marketing campaign to advertise your product. The success of this campaign depends on the amount a (CHF) spent for advertising and on the selling price p (CHF) of your product. Specifically, you determine that you will sell

$$3,000 + 4\sqrt{a} - 20\,p$$

units of your product if you spend a (CHF) for advertising and the selling price of your product is p (CHF).

Determine a and p that maximize your profit (i.e., revenues from selling minus costs, included the cost of advertising).

Exercise 1

(c) (additional space for your solution)

Exercise 1

(d) (10 points)
A rectangle R has its vertices at the points $(0,0)$, $(x,0)$, $(0,y)$, and (x,y), where $x,y>0$. Moreover, the distance between the point (x,y) and the point $(a,0)$ corresponds to 4, where $a \in (4,8)$.

Determine the candidate points (x,y) such that the area of R is extreme (maximal or minimal).

Remarks:

(1) It is not required to verify if the founded candidate point(s) do indeed generate a maximal or minimal area for R.

(2) The distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ corresponds to $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Therefore, $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

(3) No need to simply the terms of the final solution.

Exercise 1

(d) (additional space for your solution)

Part II: Multiple-choice Question (64 points)

General instructions for multiple-choice questions:

(i) The solution must be reported in the multiple-choice solution form, which is distributed together with the exam. Only the answers reported in the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility can be marked.

(iii) When two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions.

Exercise 2 (32 points)

Question 1 (3 points)

The function $f(x, y) = x$ has under the constraint $\varphi(x, y) = \frac{x^2}{36} + \frac{(y-3)^2}{16} = 1$ its maximum at

(a) $P = (-6, 3)$.

(b) $P = (5, -3)$.

(c) $P = (0, 7)$.

(d) $P = (6, 3)$.

Exercise 2

Question 2 (3 points)

A twice-differentiable function f has a local maximum at (x_0, y_0). Let g be a function defined by $g(x, y) = -f(-x, -y)$ and $D_g = D_f$. Then:

(a) g has local maximum at (x_0, y_0).

(b) g has a local minimum at (x_0, y_0).

(c) g has a local maximum $(-x_0, -y_0)$.

(d) g has a local minimum at $(-x_0, -y_0)$.

Exercise 2

Question 3 (4 points)

The function of two variables f has a local maximum at $(x_0, y_0) = (1, 2)$ under the constraint $\varphi(x, y) = x^2 + 3\,y - 7 = 0$.

The slope of the tangent line to the contour line of f at $(x_0, y_0) = (1, 2)$ corresponds to:

(a) $-\frac{3}{2}$.

(b) $-\frac{2}{3}$.

(c) $\frac{3}{2}$.

(d) $\frac{2}{3}$.

Exercise 2

Question 4 (2 points)

For a continuous function f and $a, x \in D_f$ with $a \leq x$, the integral function I is defined as $I(x) = \int_a^x f(t)\,dt$. We have:

(a) I is an antiderivative of f.

(b) $f'(x) = I(x)$.

(c) $I'(x) = f(x) - f(a)$.

(d) I is not differentiable.

Exercise 2

Question 5 (2 points)

Let F be an antiderivative of f and g a differentiable function. It follows that:

(a) $\int f(g(x))\,dx = F(x) + C, \quad C \in \mathbb{R}.$

(b) $\int f(g(x))\,dx = F(g(x)) + C, \quad C \in \mathbb{R}.$

(c) $\int f(g(x))\,f'(x)\,dx = F(g(x)) + C, \quad C \in \mathbb{R}.$

(d) $\int f(g(x))\,g'(x)\,dx = F(g(x)) + C, \quad C \in \mathbb{R}.$

Exercise 2

Question 6 (3 points)

The indefinite integral

$$\int \left[6\,x + (2\,x^2 + 1)\,e^{x^2}\right]\,dx$$

is

(a) $x^2 + x\,e^{x^2} + C, \quad C \in \mathbb{R}.$

(b) $3\,x + 2\,x\,e^{x^2} + C, \quad C \in \mathbb{R}.$

(c) $3\,x^2 + 2\,x\,e^{x^2} + C, \quad C \in \mathbb{R}.$

(d) $3\,x^2 + x\,e^{x^2} + C, \quad C \in \mathbb{R}.$

Exercise 2

Question 7 (3 points)

Given is the function f defined by

$$f(x) = \begin{cases} a\,x^2 + \frac{1}{2} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

f is a density function when:

(a) $a = \frac{1}{2}$.

(b) $a = \frac{3}{2}$.

(c) $a = \frac{5}{2}$.

(d) For no $a \in \mathbb{R}$.

Exercise 2

Question 8 (4 points)

Given is the function

$$f(x) = \begin{cases} a\,x + \frac{1}{16} & \text{for } 0 \leq x \leq 8 \\ 0 & \text{otherwise} \end{cases} .$$

The paremeter a is chosen such that f is the density function of a continuous random variable X.

The expected value $\mathbb{E}[X]$ of X is:

(a) $\mathbb{E}[X] = \frac{25}{16}$.

(b) $\mathbb{E}[X] = \frac{14}{3}$.

(c) $\mathbb{E}[X] = -\frac{5}{3}$.

(d) $\mathbb{E}[X] = \frac{7}{3}\,a$.

Exercise 2

Question 9 (2 points)

Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and $\mathbf{a}$ has length $\|\mathbf{a}\| = 3$. It follows that:

(a) $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = 0$.

(b) $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = 3$.

(c) $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}) = 9$.

(d) $\mathbf{a} \cdot (\mathbf{a} + \mathbf{b})$ cannot be determined without knowing the components of $\mathbf{a}$ and $\mathbf{b}$.

Exercise 2

Question 10 (2 points)

Given are the vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Which of the following systems is a basis of $\mathbb{R}^3$?

(a) $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

(b) $\{\mathbf{a}, \mathbf{b}, \mathbf{d}\}$.

(c) $\{\mathbf{b}, \mathbf{c}, \mathbf{e}\}$.

(d) $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$.

Exercise 2

Question 11 (2 points)

A is a 7×5 matrix. The system of linear equations $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions and the solution space has dimension 3. We have:

(a) $\mathrm{rg}(A) < \mathrm{rg}(A; \mathbf{b}) = 3$.

(b) $\mathrm{rg}(A) = \mathrm{rg}(A; \mathbf{b}) = 2$.

(c) $\mathrm{rg}(A) < \mathrm{rg}(A; \mathbf{b}) = 2$.

(d) $\mathrm{rg}(A) = \mathrm{rg}(A; \mathbf{b}) = 3$.

Exercise 2

Question 12 (2 points)

A and B are regular matrices. Moreover, A is symmetric. The expression:

$$B^T\,(AB)^T\,(B^{-1}A^{-1})^T\,B\,(AB)^{-1}$$

corresponds to:

(a) $(A^T\,B^T)^{-1}$.

(b) $(B^{-1}\,A)^T$.

(c) $(A^{-1}\,B)^T$.

(d) None of the above expressions.

Exercise 3 (32 points)

Question 1 (4 points)

X is a continuous random variable with density function

$$f(x) = \begin{cases} \frac{2}{3}\,x + \sqrt{x} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and expected value

$$\mathbb{E}\left[X\right] = \frac{28}{45}.$$

It follows that $\int_0^1 (x+1)\, f(x)\, dx$ equals to:

(a) $\frac{28}{45}$.

(b) $\frac{73}{45}$.

(c) $\frac{12}{45}$.

(d) $\frac{52}{45}$.

Exercise 3

Question 2 (4 points)

For which value of $t \in \mathbb{R}$ are the vectors $\mathbf{u} = \begin{pmatrix} t \\ t-1 \\ -6 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} t+5 \\ -1 \\ 1 \end{pmatrix}$ orthogonal and the lenght of $\mathbf{u}$ equals $\sqrt{37}$?

(a) $t = -5$.

(b) $t = 0$.

(c) $t = 1$.

(d) No $t \in \mathbb{R}$.

Exercise 3

Question 3 (3 points)

The 3×4 matrix

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$$

(a) has rank 1.

(b) has rank 2.

(c) has rank 3.

(d) has rank 4.

Exercise 3

Question 4 (5 points)

Given is the 3×3 matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have:

(a) $A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) $A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

(c) $A^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(d) A is singular.

Exercise 3

Question 5 (4 points)

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

The matrix A^2 has the eigenvalue:

(a) 2.

(b) 4.

(c) 6.

(d) 8.

Exercise 3

Question 6 (4 points)

The initial value problem

$$y_{k+1} - (1+a)\,y_k = 2\,a, \text{ where } a \neq -1,\ a \neq 0,$$
$$y_0 = 2$$

has the solution

(a) $y_k = -4\,(1+a)^k$.

(b) $y_k = 2(1+a)^k - 1$.

(c) $y_k = 4\,(1+a)^k - 2$.

(d) $y_k = 8\,(1+a)^k - 3$.

Exercise 3

Question 7 (3 points)

The general solution of the difference equation

$$3\,(y_{k+1} - y_k) + 5 = 2\,y_{k+1} - y_k + 12$$

is

(a) oscillating and convergent.

(b) oscillating and divergent.

(c) monotone and convergent.

(d) monotone and divergent.

Exercise 3

Question 8 (5 points)

The general solution of the linear difference equation

$$2\,(a+2)\,y_{k+1} - 2\,y_k + 2\,(a^2-4) = 0, \quad k = 0, 1, 2, \ldots,$$

where $a \in \mathbb{R} \setminus \{-2, -1\}$, monotonically converges to 0 if and only if

(a) $a > -2$.

(b) $a > -1$.

(c) $a = 2$.

(d) $a = 1$.

Exams Assessment Level: Spring Semester 2018

2’202 Mathematics B

Multiple-choice answer sheet

Exercise 2 (32 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Question 9: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Question 10: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Question 11: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Question 12: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Exams Assessment Level: Spring Semester 2018

2'202 Mathematics B

Multiple-choice answer sheet

Exercise 3 (32 points)

Question 1: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (5 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (5 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐