Mathematics A
Master Solutions Exam Autumn Semester 2018

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Part I: Open questions

Exercise 1

(a1) (**6 points**).
A market equilibrium $(p^\star, q^\star)$ satisfies the property that market demand and market supply coincide to $q^\star$ at the price $p^\star$; mathematically:

$$q^\star = q_s(p^\star) = q_d(p^\star) \Leftrightarrow q^\star = 4\,e^{p^\star-1} = 70 - 15\,p^\star - 1.5\,(p^\star)^2.$$

It follows that $(p^\star, q^\star)$ is a market equilibrium if and only if:

$$4\,e^{p^\star-1} - 70 + 15\,p^\star + 1.5\,(p^\star)^2 = 0$$

and $q^\star = 4\,e^{p^\star-1} = 70 - 15\,p^\star - 1.5\,(p^\star)^2$.

Let $f(p) = 4\,e^{p-1} - 70 + 15\,p + 1.5\,p^2$, then there is a unique market equilibrium $(p^\star, q^\star)$ with $p^\star \in [2,3]$ if and only if function f has a unique root on $[2,3]$, i.e., there is a unique $p^\star \in [2,3]$ with $f(p^\star) = 0$.

Existence of $p^\star$: Because f is continuous on $[2,3]$ and $f(2) = 4\,e - 34 \approx -23.127 < 0$, $f(3) = 4\,e^2 - 11.5 \approx 18.056 > 0$, then according to Bolzano's Nullstellensatz, there exists at least one $p^\star \in [2,3]$ with $f(p^\star) = 0$.

Uniqueness of $p^\star$: Because

$$f'(p) = 4\,e^{p-1} + 3\,p + 15 \geq f'(2) = 4\,e + 21 \approx 31.873 > 0,$$

for all $p \in [2,3]$, then f is strictly increasing on $(2,3)$. Therefore, $p^\star$ with $f(p^\star) = 0$ must be unique on $[2,3]$.

(a2) (**6 points**).
To solve the equilibrium equation $f(p) = 0$ on $[2,3]$, we approximate f with its second order Taylor polynomial P_2 at $p_0 = 1$. We have:

$$P_2(p) = f(1) + f'(1)\,(p-1) + \frac{1}{2}\,f''(1)\,(p-1)^2$$

where

$$f(1) = 4 + 1.5 + 5 - 70 = -49.5,$$

$$f'(p) = 4\,e^{p-1} + 3\,p + 15 \Rightarrow f'(1) = 4 + 3 + 15 = 22,$$

and

$$f''(p) = 4\,e^{p-1} + 3 \Rightarrow f''(1) = 4 + 3 = 7.$$

It follows that:

$$P_2(p) = -49.5 + 22\,(p-1) + \frac{7}{2}\,(p-1)^2 = \frac{7}{2}\,p^2 + 15\,p - 68.$$

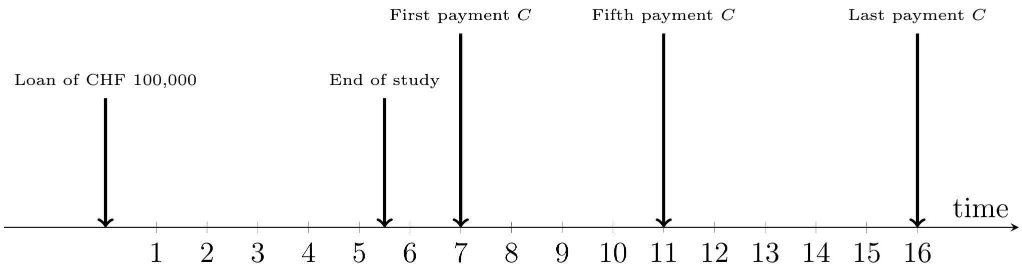
Therefore,

$$P_2(p) = 0 \Leftrightarrow \frac{7}{2}\,p^2 + 15\,p - 68 = 0 \Leftrightarrow p = \frac{-15 \pm \sqrt{15^2 - 4\,\frac{7}{2}\,(-68)}}{2\,\frac{7}{2}} \Leftrightarrow p = \frac{-15 \pm \sqrt{1177}}{7}.$$

Because $p_1 = \frac{-15+\sqrt{1177}}{7} \approx 2.7582 \in [2,3]$ and $p_2 = \frac{-15-\sqrt{1177}}{7} \approx -7.0439 \notin [2,3]$, then the unique equilibrium price in $[2,3]$ is approximately $p^\star = 2.7582$.

(b) (**10 points**).

(b1)



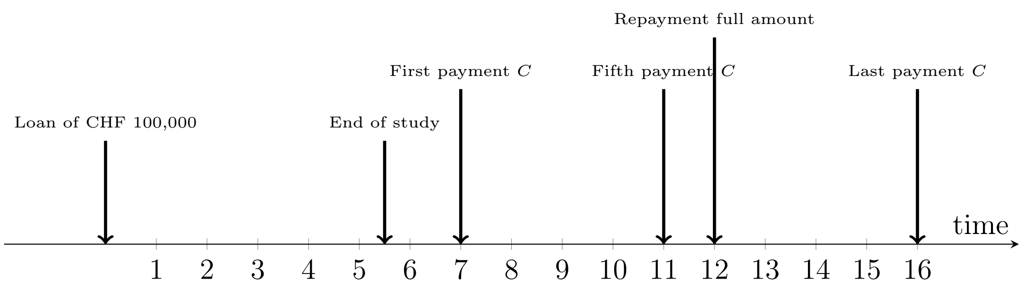
(b2) The debt A_6 after $n_1 = 6$ years corresponds to:

$$A_6 = A_0\,(1+i)^{n_1} = 100,000\,(1+0.08)^6 \approx 158,687.45 \text{ (CHF)}.$$

(b3) The ten yearly payments C at the end of each year for 10 years, starting in year 6, correspond to an annuity immediate with $P = A_6$ (initial value), $n_2 = 10$ and $i = 8\%$. We have:

$$\begin{aligned} C &= \frac{i}{(1+i)^{n_2} - 1}\,(1+i)^{n_2}\,P \\ &= \frac{i\,P}{1-(1+i)^{-n_2}} \\ &= \frac{i\,A_6}{1-(1+i)^{-n_2}} \\ &= \frac{0.08 \cdot 158,687.45}{1-(1+0.08)^{-10}} \\ &\approx 23,649.10 \text{ (CHF)}. \end{aligned}$$

(b4)



(b4) The outstanding debt at the beginning of year 12 corresponds to the value $A_{12} = A_0\,(1+i)^{12} = 100,000\,(1+0.08)^{12}$ of the debt in year 12 in case of no yearly payments, *minus* the value $C\,\sum_{k=1}^{5}(1+0.08)^k$ in year 12 of the 5 yearly payments at the *end* of years 6, 7, 8, 9, and 10 (i.e., the first payment is compounded for 5 years, the second payment for 4 years, the third payment for 3 years, the fourth payment for 2 years, and the last payment for 1 year). We have:

$$\begin{aligned}\text{Outstanding debt} &= A_0\,(1+i)^{12} - C\sum_{k=1}^{5}(1+0.08)^k \\ &= 100,000\,(1+0.08)^{12} - 23,649.10 \cdot 0.08 \cdot \frac{1.08^5 - 1}{0.08} \\ &\approx 101,977.95(\text{CHF}).\end{aligned}$$

It follows that Nina will not be able to repay the full outstanding debt using her inheritance. She will have to pay an additional 1,977.95 CHF to fully cancel her debt.

(c) (**5 points**).
For $t = 0, 1, 2, \ldots$ let $G(t)$ be the percentage of identical elements in the Finnish and the Hungarian languages t years *after* their separation.

Because $G(0) = 100\%$ and $\frac{G(t+2000)}{G(t)} = \frac{1}{2}$ for all t (i.e., the percentage of identical elements halves every 2000 years), we have:

$$G(t) = 100\% \left(\frac{1}{2}\right)^{\frac{t}{2000}}.$$

To determine t, we apply the estimated $G(t)$ that lies between 21% and 27%:

$$\begin{aligned}& 21\% \le 100\% \left(\frac{1}{2}\right)^{\frac{t}{2000}} \le 27\% \\ \Leftrightarrow\ & \frac{\ln(0.27)}{\ln(0.5)}\,2000 \le t \le \frac{\ln(0.21)}{\ln(0.5)}\,2000 \\ \Leftrightarrow\ & 3778 \le t \le 4503.\end{aligned}$$

According to the model above, the Finnish and the Hungarian languages have separated between 3,800 and 4,500 years ago.

(d) (**9 points**).

(d1) Because the present value of consumption and income must coincide, we have:

$$c_1 + \frac{c_2}{(1+0.08)} = 3,000 + \frac{7,000}{(1+0.08)}.$$

Multiplying both sides by 1.08 we obtain:

$$c_2 = 7,000 - 1.08\,(c_1 - 3,000) = 10,240 - 1.08\,c_1.$$

(d2) Replacing c_1 with $c_2 = 10,240 - 1.08\,c_1$ in the formula for utility u we obtain:

$$u = \ln(c_1) + \frac{1}{1.2}\ln(c_2) = \ln(c_1) + \frac{1}{1.2}\ln(10,240 - 1.08\,c_1) = v(c_1).$$

(d3) To maximize utility u, the student has to equivalently maximize function v from (d2). To do so, we solve the first order condition $v'(c_1) = 0$. We have:

$$v'(c_1) = \frac{1}{c_1} - \frac{1.08}{1.2}\,\frac{1}{10,240 - 1.08\,c_1} = 0 \Leftrightarrow c_1 = \frac{1.2}{1.0.8}\,(10,240 - 1.08\,c_1) \Leftrightarrow 2.2\,c_1 = \frac{1.2 \cdot 10,240}{1.08}.$$

It follows that $c_1^\star = \frac{1.2 \cdot 10,240}{2.2 \cdot 1.08} \approx 5,171.70(\text{CHF})$ is a stationary point of v. To check if c_1 is also a maximum of v, we compute the second order derivative v''. We have:

$$v'' = -\frac{1}{c_1^2} - \frac{1.08^2}{1.2}\,\frac{1}{(10,240 - 1.08\,c_1)^2} < 0.$$

Therefore, v is overall concave and $c_1^\star$ is a global maximum of v. Consequently, $c_2^\star = 10,240 - 1.08\,c_1^\star \approx 4,654.55(\text{CHF})$. It follows that the student has to borrow 2,171.70 CHF to finance her consumption at the time $t = 1$ and repay 2,345.45 CHF at time $t = 2$ to cancel her debt.

Part II: Multiple-choice questions

Exercise 2

1. Answer is (c). Let $A =$ "Albert participate" and $B =$ "Beatrice participate". It follows that the statement "If Albert participates, then Beatrice participates too" corresponds to $A \Rightarrow B$ and is known to be true.

We have: (a): $B \Rightarrow A$; (b): $\neg A \Rightarrow \neg B$; (c): $\neg B \Rightarrow \neg A$.

The following truth table applies:

	A	T	T	F	F
	B	T	F	T	F
	$\neg A$	F	F	T	T
	$\neg B$	F	T	F	T
	$A \Rightarrow B$	T	F	T	T
(a)	$B \Rightarrow A$	T	T	F	T
(b)	$\neg A \Rightarrow \neg B$	T	T	F	T
(c)	$\neg B \Rightarrow \neg A$	T	F	T	T

Therefore, (c) is true if and only if the original statement is also true. Indeed, if Beatrice does not participate, then Albert also does not participate. Otherwise, if Albert would participate, then the true statement $A \Rightarrow B$ would imply that also Beatrice participates, which is a contradiction.

2. Answer is (a). If $\{a_n\}_{n\in\mathbb{N}}$ is monotone increasing and bounded, then also $\{b_n\}_{n\in\mathbb{N}}$ is monotone increasing and bounded. Because any sequence that is monotone and bounded is also convergent, then $\{b_n\}_{n\in\mathbb{N}}$ is convergent.

3. Answer is (c). The effective interest i_{eff} satisfies the following condition:

$$1,000,000\left(1+\frac{2\%}{12}\right)^{12} = 1,000,000\,(1+i_{\text{eff}}).$$

Indeed, i_{eff} corresponds to the interest rate that would deliver the same amount with annual compounding as in the case of monthly compounding with annual rate 2%.

It follows that:

$$i_{\text{eff}} = \left(1+\frac{2\%}{12}\right)^{12} - 1 \approx 2.018\%.$$

4. Answer is (c). We have:

$$y = -\frac{3}{x^5} + 2 \Leftrightarrow \frac{3}{x^5} = 2 - y \Leftrightarrow x^5 = \frac{3}{2-y} \Leftrightarrow x = \sqrt[5]{\frac{3}{2-y}}.$$

Therefore, $f^{-1}(x) = \sqrt[5]{\frac{3}{2-x}}$ is defined for $x \neq 2$, i.e., $D_{f^{-1}} = \mathbb{R} \setminus \{2\}$.

Alternatively, one could also use that $D_{f^{-1}} = R_f$. Because f takes all values a part 2, then $R_f = \mathbb{R} \setminus \{2\}$.

5. Answer is (b). We have:

$$n^{\log_m(k)} = \left(e^{\ln(n)}\right)^{\frac{\ln(k)}{\ln(m)}} = \left(e^{\ln(k)}\right)^{\frac{\ln(n)}{\ln(m)}} = k^{\log_m(n)}.$$

6. Answer is (b). Because $\sin(x) = 0$ for $x \in \{-\pi, 0, \pi\}$, then $\{-\pi, 0, \pi\} \notin D_f$, and thus f is discontinuous at these points (which are poles). By contrast, f is continuous at all $x_0 \in D_f \cap (-2\,\pi, 2\,\pi)$.

7. Answer is (d). We have $f'(x) = \frac{1}{3} + 3\,\frac{e}{x^4}$. Because $3\,\frac{e}{x^4} > 0$ for all $x \in D_f$, then $f'(x) > \frac{1}{3}$ for all $x \in D_f$. Moreover, each value y in $\left(\frac{1}{3}, \infty\right)$ is taken by f' because $f'(x) = y \Leftrightarrow x = \sqrt[4]{\frac{e}{3}\,\left(y - \frac{1}{3}\right)}$.

8. Answer is (b). Let f be an even function. It follows that $f(x) = f(-x)$. Therefore, $f'(x) = -f'(-x)$, i.e., f' is an odd function. Therefore, (b) is false. (a) is true, because f is differentiable, and thus continuous. (c) is true, as we just proved. (d) is true, because f is differentiable, then $f' > 0$ implies that f is strictly increasing.

9. Answer is (d).

We have:

(a) $d(a\,f + b\,g)(x) = (a\,f'(x) + b\,g'(x))\,dx = a\,f'(x)\,dx + b\,g'(x)\,dx = a\,df(x) + b\,dg(x)$.

(b) $d(f\,g)(x) = (f'(x)\,g(x) + f(x)\,g'(x))\,dx = f'(x)\,dx\,g(x) + f(x)\,g'(x)\,dx = g(x)\,df(x) + f(x)\,dg(x)$.

(c) $d(f \circ g)(x) = (f'(g(x))\,g'(x))\,dx = f'(g(x))\,g'(x)\,dx = f'(g(x))\,dg(x)$.

(d) $d\left(\frac{f}{g}\right)(x) = \left(\frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}\right)\,dx = \frac{f'(x)\,dx}{g(x)} - \frac{f(x)}{g^2(x)}\,g'(x)\,dx = \frac{df(x)}{g(x)} - \frac{f(x)}{g^2(x)}\,dg(x)$.

10. Answer is (a). The quadratic function $g(x) = x^2 - 2\,x - 24 = (x+4)\,(x-6)$ is negative for $x \in (-4, 6)$ and has a global minimum at $x_0 = -\frac{-2}{2} = 1$. Because $f(x) = |g(x)|$, then $f(x) = g(x)$ for $x \notin (-4, 6)$ and $f(x) = -g(x)$ for $x \in (-4, 6)$. I.e., $x_0 \in (-4, 6)$ becomes a local maximum of f (not a global maximum, as $f(x)$ diverges to infinity for $x \to \pm\infty$).

11. Answer is (c). By definition of the Taylor polynomial P, $P^{(k)}(0) = f^{(k)}(0)$ for $k = 0, 1, 2, 3$, where $P^{(0)}(0) = P(0)$ and $f^{(0)}(0) = f(0)$. We have:

$$P(0) = 0, P'(0) = 2, P''(0) = 10, P'''(0) = -6.$$

Therefore, only (c) is true.

12. Answer is (c). Euler's relation implies:

$$\varepsilon_{f,x}(x, y) + \varepsilon_{f,y}(x, y) = k,$$

where k is the degree of homogeneity of f. It follows:

$$\varepsilon_{f,x}(x, y) = k - \varepsilon_{f,y}(x, y) = 1.8 - (x + y - 1.2) = -x - y + 3.$$

Exercise 3

1. Answer is (d). We have:

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos(x)} \overset{\text{de l'Hôpital}}{=} \lim_{x\to 0}\frac{e^x-e^{-x}}{\sin(x)} \overset{\text{de l'Hôpital}}{=} \lim_{x\to 0}\frac{e^x+e^{-x}}{\cos(x)} = 2.$$

Therefore, (d) is correct.

2. Answer is (b). Because $x^3-27=(x-3)\,(x^2+3\,x+9)$, we then have:

$$\begin{aligned}
\lim_{x\to 3}\left(\frac{1}{x-3}-\frac{27}{x^3-27}\right) &= \lim_{x\to 3}\frac{(x^2+3\,x+9)-27}{(x-3)\,(x^2+3\,x+9)}\\
&= \lim_{x\to 3}\frac{x^2+3\,x-18}{(x-3)\,(x^2+3\,x+9)}\\
&= \lim_{x\to 3}\frac{(x-3)\,(x+6)}{(x-3)\,(x^2+3\,x+9)}\\
&= \lim_{x\to 3}\frac{x+6}{x^2+3\,x+9}\\
&= \frac{9}{27}=\frac{1}{3}.
\end{aligned}$$

3. Answer is (b). A natural number $n\in\mathbb{N}$ with s places in the decimal system can be written as $n=x\,10^{s-1}$, where $x\in[1,10)$. For example, 123 has 3 places and we have $123=1.23\cdot 10^2$. It follows that:

$$\log_{10}(n)=\log_{10}(x)+(s-1)\Leftrightarrow s=\log_{10}(n)+1-\log_{10}(x).$$

Because $s\in\mathbb{N}$ and $\log_{10}(x)\in[0,1)$, then s is the *largest* integer smaller than or equal to $\log_{10}(n)+1$.

We have:

$$\begin{aligned}
\log_{10}(p)+1 &= \log_{10}(p+1)+1\\
&= \log_{10}\left(2^{77,232,917}\right)+1\\
&= 77,232,910\,\frac{\ln(2)}{\ln(10)}+1\\
&\approx 23,249,424.7+1\\
&= 23,249,425.7.
\end{aligned}$$

Because the largest integer lower than or equal to 23,249,425.6696 is 23,249,425, then p has 23,249,425 places.

4. Answer is (d). For function f,

$$\begin{aligned}
(x,y)\in D_f &\Leftrightarrow 4\,x^2+4\,y^2-36\geq 0 \text{ and } x^2-y-2>0\\
&\Leftrightarrow x^2+y^2\geq 9=3^2 \text{ and } y<x^2-2.
\end{aligned}$$

The inequality $x^2+y^2\geq 9=3^2$ determines the area outside the circle with centre $(0,0)$ and radius 3. The inequality $y<x^2-2$ determine the area below the parabola $y=x^2-2$ (without the parabola itself).

5. Answer is (d). We have:

$$(x,y) \in D_f \Leftrightarrow 0 < \sqrt{e} - x^2 - y^2 \leq \sqrt{e} \Leftrightarrow -\infty < \ln(\sqrt{3} - x^2 - y^2) \leq \frac{1}{2} \Leftrightarrow f(x,y) \in \left(-\infty, \frac{1}{2}\right].$$

6. Answer is (c). Because in any neighborhood U of $(1,1)$ that is small enough, we have $y+1>0$ and $x-2<0$, then

$$f(x,y) = g(x,y) = \ln\left(\frac{y+1}{2-x}\right) = \ln(y+1) - \ln(2-x)$$

for all $(x,y) \in U$. The implicit function theorem implies that:

$$m = -\frac{f_x(1,1)}{f_y(1,1)} = -\frac{g_x(1,1)}{g_y(1,1)} = -\frac{\frac{1}{2-x}}{\frac{1}{y+1}}\Big|_{(x,y)=(1,1)} = -2.$$

7. Answer is (a). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\,x, \lambda\,y) &= \ln\left((\lambda\,x)^2\,\sqrt{\lambda\,y} + \sqrt[4]{(\lambda\,x)^3\,(\lambda\,y)^7}\right) - \frac{5}{2}\,\ln(\lambda\,x) \\
&= \ln\left(\lambda^{\frac{5}{2}}\,x^2\,\sqrt{y} + \lambda^{\frac{10}{4}}\,\sqrt[4]{x^3\,y^7}\right) - \frac{5}{2}\ln(x) - \frac{5}{2}\,\ln(\lambda) \\
&= \ln\left(\lambda^{\frac{5}{2}}\left(x^2\,\sqrt{y} + \sqrt[4]{x^3\,y^7}\right)\right) - \frac{5}{2}\ln(x) - \frac{5}{2}\,\ln(\lambda) \\
&= \ln\left(\lambda^{\frac{5}{2}}\right) + \ln\left(x^2\,\sqrt{y} + \sqrt[4]{x^3\,y^7}\right) - \frac{5}{2}\ln(x) - \frac{5}{2}\,\ln(\lambda) \\
&= \frac{5}{2}\,\ln(\lambda) + \ln\left(x^2\,\sqrt{y} + \sqrt[4]{x^3\,y^7}\right) - \frac{5}{2}\ln(x) - \frac{5}{2}\,\ln(\lambda) \\
&= \ln\left(x^2\,\sqrt{y} + \sqrt[4]{x^3\,y^7}\right) - \frac{5}{2}\ln(x) \\
&= f(x,y) \\
&= \lambda^0\,f(x,y).
\end{aligned}$$

It follows that f is homogeneous of degree 0.

8. Answer is (b). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\,x, \lambda\,y) &= 7\,(\lambda\,x)\,\sqrt{(\lambda\,y)^a} + 3\,(\lambda\,y)^2\,\sqrt[4]{(\lambda\,x)^a\,(\lambda\,y)^b} - (\lambda\,x)^2\,(\lambda\,y)^{0.2} \\
&= \lambda^{1+\frac{1}{2}\,a}\,7\,x\,\sqrt{y^a} + \lambda^{2+\frac{1}{4}\,a+\frac{1}{4}\,b} \cdot 3\,y^2\,\sqrt[4]{x^a\,y^b} - \lambda^{2.2}\,x^2\,y^{0.2}.
\end{aligned}$$

Therefore, f is homogeneous if and only if

$$1 + \frac{1}{2}\,a = 2.2 = 2 + \frac{1}{4}\,a + \frac{1}{4}\,b \Leftrightarrow a = 2.4 \text{ and } b = -1.6.$$