Mathematics A
Master Solutions Alternative Date Exam
Autumn Semester 2017

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Part I: Open questions

Exercise 1

(a1) (**4 points**).
The function f is defined if and only if the following three conditions apply:

(i) $x+5 \geq 0$ because of the $\sqrt{\cdot}$ function;

(ii) $2-e^{-3\sqrt{x+5}} \neq 0$ because division by 0 is not allowed;

(iii) $\frac{1}{2-e^{-3\sqrt{x+5}}} > 0$ because of the $\ln(\cdot)$ function.

Clearly, condition (ii) holds when condition (iii) holds. Moreover, condition (iii) holds when condition (i) holds, because $2-e^{-3\sqrt{x+5}} \geq 1$ when $x+5 \geq 0$. Therefore, we only have to solve condition (i) and this implies $x \geq -5$. It follows that:

$$D_f = [-5, \infty).$$

For the range R_f of f we have:

$$\begin{aligned} x \in D_f \quad &\Leftrightarrow \quad x \geq -5 \\ &\Leftrightarrow \quad 1 \leq 2-e^{-3\sqrt{x+5}} < 2 \\ &\Leftrightarrow \quad \frac{1}{2} < \frac{1}{2-e^{-3\sqrt{x+5}}} \leq 1 \\ &\Leftrightarrow \quad \ln\left(\frac{1}{2}\right) < \ln\left(\frac{1}{2-e^{-3\sqrt{x+5}}}\right) \leq 0. \end{aligned}$$

It follows that:

$$R_f = \left(\ln\left(\frac{1}{2}\right), 0\right] = \left(-\ln(2), 0\right].$$

(a2) (**3 points**).
We apply the following result: If f is differentiable and $f'(x) > 0$ (< 0) for all $x \in (a,b)$, then f is strictly increasing (decreasing) on $[a,b]$.

Because in our case

$$f(x) = \ln\left(\frac{1}{2-e^{-3\sqrt{x+5}}}\right) = -\ln\left(2-e^{-3\sqrt{x+5}}\right),$$

then

$$f'(x) = \frac{-1}{2-e^{-3\sqrt{x-5}}}\left(-e^{-3\sqrt{x+5}}\right)\left(\frac{-3}{2\sqrt{x+5}}\right).$$

Because all three factors are strictly negative on the domain of f, then $f'(x) < 0$ for all $x \in (-5, \infty)$ and thus f is strictly decreasing on its domain $D_f = [-5, \infty)$.

(a3) (**3 points**).
For $y \in R_f = (-\ln(2), 0]$ we have:

$$\begin{aligned} y = -\ln\left(2 - e^{-3\sqrt{x+5}}\right) &\Leftrightarrow e^{-y} = 2 - e^{-3\sqrt{x+5}} \\ &\Leftrightarrow 2 - e^{-y} = e^{-3\sqrt{x+5}} \\ &\Leftrightarrow \ln\left(2 - e^{-y}\right) = -3\sqrt{x+5} \\ &\Leftrightarrow -\frac{1}{3}\ln\left(2 - e^{-y}\right) = \sqrt{x+5} \\ &\Leftrightarrow \frac{1}{9}\left(\ln\left(2 - e^{-y}\right)\right)^2 = x + 5 \\ &\Leftrightarrow \frac{1}{9}\left(\ln\left(2 - e^{-y}\right)\right)^2 - 5 = x. \end{aligned}$$

Because $D_{f^{-1}} = R_f = (-\ln(2), 0]$ and $R_{f^{-1}} = D_f = [-5, \infty)$ it follows that:

$$f^{-1} : (-\ln(2), 0] \to [-5, \infty), \quad x \mapsto y = f^{-1}(x) = \frac{1}{9}\left(\ln\left(2 - e^{-x}\right)\right)^2 - 5.$$

(b) (**6 points**).
The borrowed amount of $P = 1,000,000$ CHF has a final value in 25 years corresponding to:

$$A = P(1 + i_1)^{10}(1 + i_2)^{15}$$

where $i_1 = 1\%$ and $i_2 = 2\%$. Moreover, the constant payments $C_1^I = 25,000$ CHF at end of each year for 10 years correspond to a 10-year annuity immediate with end value after 10 years given by:

$$A_{1,10} = C_1^I \frac{(1 + i_1)^{10} - 1}{i_1}.$$

The amount $A_{1,10}$ compounded for additional 15 years at the interest rate i_2 corresponds to:

$$A_{1,10}(1 + i_2)^{15} = C_1^I \frac{(1 + i_1)^{10} - 1}{i_1}(1 + i_2)^{15}.$$

Finally, the constant payments of C_2^I CHF at the end of each year for the *next* 15 years correspond to a 15-year annuity immediate with end value at the end of the 25 years period given by

$$A_{2,15} = C_2^I \frac{(1 + i_2)^{15} - 1}{i_2}.$$

Because the value of the debt after 25 years from its issue must correspond to 200,000 CHF, the following conditions must hold for the only unknown C_2^I:

$$200,000 = \underbrace{P\,(1+i_1)^{10}\,(1+i_2)^{15}}_{\text{value in 25 year of the initial debt}} - \underbrace{C_1^I\,\frac{(1+i_1)^{10}-1}{i_1}\,(1+i_2)^{15}}_{\text{value in 25 years of the payments done over first 10 years}} - \underbrace{C_2^I\cdot\frac{(1+i_2)^{15}-1}{i_2}}_{\text{value in 25 years of the payments done over last 15 years}}$$

$$\Leftrightarrow\quad C_2^I\,\frac{1}{(1+i_1)^{10}}\,\frac{1-(1+i_2)^{-15}}{i_2} = P - \frac{200,000}{(1+i_1)^{10}\,(1+i_2)^{15}} - C_1^I\,\frac{1-(1+i_1)^{-10}}{i_1}$$

$$\Leftrightarrow\quad C_2^I = \left[P - \frac{200,000}{(1+i_1)^{10}\,(1+i_2)^{15}} - C_1^I\,\frac{1-(1+i_1)^{-10}}{i_1}\right]\frac{(1+i_1)^{10}\,i_2}{1-(1+i_2)^{-15}}$$

$$\Leftrightarrow\quad C_2^I = \left[1,000,000 - \frac{200,000}{(1+1\%)^{10}\,(1+2\%)^{15}} - 25,000\,\frac{1-(1+1\%)^{-10}}{1\%}\right]\frac{(1+1\%)^{10}\cdot 2\%}{1-(1+2\%)^{-15}}$$

$$\Leftrightarrow\quad C_2^I \approx 54,047.00 \quad \text{(CHF)}.$$

(c) (**4 points**).
We have:

$$\lim_{x\to 0}\frac{3^x-2^x}{x^2-x} \overset{\text{de l'Hôpital}}{=} \lim_{x\to 0}\frac{3^x\,\ln(3)-2^x\,\ln(2)}{2\,x-1} = -(\ln(3)-\ln(2)) = \ln(2)-\ln(3) = \ln\left(\frac{2}{3}\right).$$

(d) (**6 points**).
Let a_n be the amount of data produced in day n. We have:

$$\begin{aligned} a_0 &= 2.5 \cdot 10^{18}, \\ a_n &= a_{n-1}\,(1+5\%), \quad n = 1, 2, 3, \ldots \end{aligned}$$

i.e., $\{a_n\}_{n\in\mathbb{N}}$ is a geometric sequence with $a_0 = 2.5 \cdot 10^{18}$ and $q = 1.05$.

The questions asks about the number of years needed to reach a total data production of $45 \cdot 10^3 \cdot 10^{18}$ bytes. First all we compute the requested period in days. Let N be the number of days needed to reach a total data production of $45 \cdot 10^3 \cdot 10^{18}$ bytes. The condition on N is that

$$\sum_{n=0}^{N} a_n = 45 \cdot 10^3 \cdot 10^{18}.$$

Because $\{a_n\}_{n\in\mathbb{N}}$ is a geometric sequence, we have:

$$\sum_{n=0}^{N} a_n = a_0 \frac{1-q^{N+1}}{1-q} = 2.5 \cdot 10^{18} \frac{1-1.05^{N+1}}{1-1.05}.$$

Therefore,

$$\sum_{n=0}^{N} a_n = 45 \cdot 10^3 \cdot 10^{18} \Leftrightarrow 10^{18} \frac{1-1.05^{N+1}}{1-1.05} = 45 \cdot 10^3 \cdot 10^{18} \Leftrightarrow 1 + 0.05 \cdot \frac{45 \cdot 10^3}{2.5} = 1.05^{N+1}.$$

It follows that:

$$N = \frac{\ln\left(1 + 0.05 \cdot \frac{45\cdot 10^3}{2.5}\right)}{\ln(1.05)} - 1 \approx 138.44 \quad (days).$$

This corresponds to approximatively 0.38 years.

Exercise 2

(a1) (**5 points**).
The third order Taylor polynomial of f at x_0 is given by:

$$P_2(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2}\,(x - x_0)^2 + \frac{f'''(x_0)}{3!}\,(x - x_0)^3.$$

With $x_0 = 0$ we have:

$$\begin{aligned} f(x_0) &= f(0) = \sin(0) = 0 \\ f'(x) &= \cos(x) \Rightarrow f'(x_0) = f'(0) = \cos(0) = 1 \\ f''(x) &= -\sin(x) \Rightarrow f''(x_0) = f''(0) = -\sin(0) = 0 \\ f'''(x) &= -\cos(x) \Rightarrow f'''(x_0) = f'''(0) = -\cos(0) = -1. \end{aligned}$$

It follows that:

$$P_3(x) = x - \frac{1}{6}\,x^3.$$

Therefore,

$$a_k = \sin\left(1 + \left(\frac{1}{4}\right)^k\right) \approx P_3\left(\left(\frac{1}{4}\right)^k\right) = \left(\frac{1}{4}\right)^k - \frac{1}{6}\left(\frac{1}{4}\right)^{3k} = \left(\frac{1}{4}\right)^k - \frac{1}{6}\left(\frac{1}{64}\right)^k.$$

It follows that:

$$\begin{aligned} \sum_{k=1}^{\infty} a_k &\approx \sum_{k=1}^{\infty} \left[\left(\frac{1}{4}\right)^k - \frac{1}{6}\left(\frac{1}{64}\right)^k\right] \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k - \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{1}{64}\right)^k \\ &= \frac{1}{4}\,\frac{1}{1-\frac{1}{4}} - \frac{1}{6}\cdot\frac{1}{64}\,\frac{1}{1-\frac{1}{64}} \\ &= \frac{1}{3} - \frac{1}{378} \\ &= \frac{125}{378} \\ &\approx 0.33. \end{aligned}$$

(a2) (**4 points**).
According to Taylor's Theorem we have:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}\,x^4$$

where $\xi \in [0, x]$.

We have:

$$f^{(4)}(x) = \sin(x).$$

Therefore, for any $x \in \mathbb{R}$,

$$|R_3(x)| = \frac{\left|f^{(4)}(\xi)\right|}{4!}\,|x|^4 = \frac{1}{24}\underbrace{|\sin(\xi)|}_{\leq 1}\,|x|^4 \leq \frac{1}{24}\,|x|^4.$$

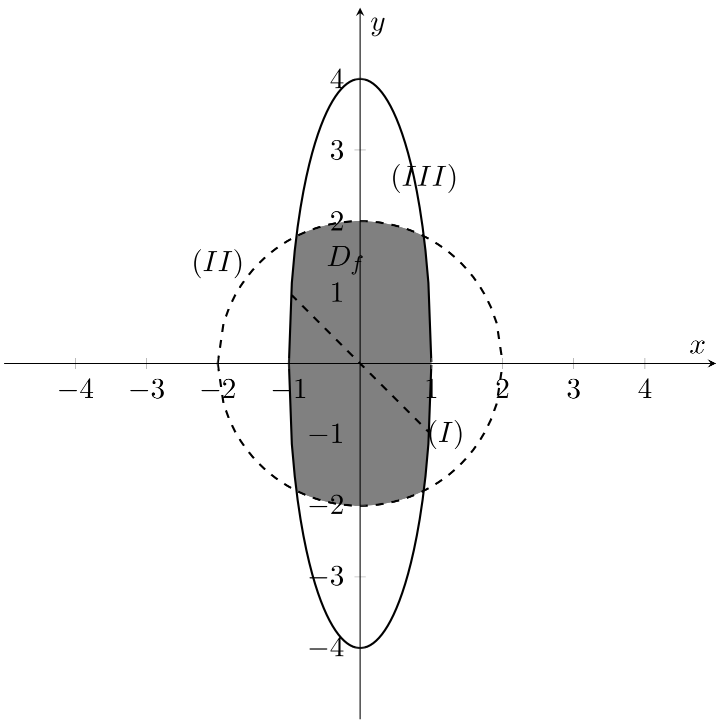
If follows that:

$$\sum_{k=1}^{\infty} R_3\left(\left(\frac{1}{4}\right)^k\right) \leq \sum_{k=1}^{\infty}\frac{1}{24}\left[\left(\frac{1}{4}\right)^k\right]^4 = \frac{1}{24}\sum_{k=1}^{\infty}\left(\frac{1}{256}\right)^k = \frac{1}{24}\cdot\frac{1}{256}\cdot\frac{1}{1-\frac{1}{256}} = \frac{1}{6120}.$$

(b) (**4 points**).
We have:

$$x \in D_f \Leftrightarrow \begin{cases} x+y \neq 0 \\ 4-x^2-y^2 > 0 \\ 16-16\,x^2-y^2 \geq 0 \end{cases} \Leftrightarrow \begin{cases} x \neq -y \ (I) \\ x^2+y^2 < 2^2 \ (II) \\ \frac{(x-0)^2}{1^2}+\frac{(y-0)^2}{4^2} \leq 1 \ (III) \end{cases}.$$

This corresponds to the area inside or on the ellipse (III) with centre $(0,0)$ and semi-axis $a=1$ and $b=4$ intersected with the area inside the circle (II) with centre $(0,0)$ and radius 2, and then excluded all points on the line $y=-x$ (I). The line $y=-x$ intersects the ellipse when $x^2+\frac{x^2}{16}=1$, i.e., $x^2=\frac{16}{17}$ or $x=\pm\frac{4}{\sqrt{17}}$. The following figure illustrates the domain of f:



(c) (**5 points**).
The parameters α, β are such that the following two conditions hold:

(i) The point $(K_0, A_0) = (2,1)$ belongs to isoquant $P(K,A) = 144$, i.e., $P(2,1) = 144$. It follows that:

$$(2 \cdot 2^{\alpha} + 4)^2 = 144 \Leftrightarrow 2 \cdot 2^{\alpha} + 4 = 12 \Leftrightarrow \alpha = 2.$$

(ii) The substitution rate $\frac{dA}{dK}$ at $(K_0, A_0) = (2,1)$ corresponds to -1. We have:

$$-1 = \left.\frac{dA}{dK}\right|_{(K_0,A_0)=(2,1)} = -\frac{P_K(K_0,A_0)}{P_A(K_0,A_0)} = -\frac{2\,(2\,K_0^{\alpha} + 4\,A_0^{\beta})\,\alpha\,2\,K_0^{\alpha-1}}{2\,(2\,K_0^{\alpha} + 4\,A_0^{\beta})\,4\,\beta\,A_0^{\beta-1}} \overset{\alpha=2,(K_0,A_0)=(2,1)}{=} -\frac{2}{\beta}.$$

It follows that:

$$\beta = 2.$$

(d) (**6 points**).
Because f is homogeneous of degree $r+2$ and g is homogeneous of degree $3-r$, then

$$h(\lambda\,x, \lambda\,y) = f(\lambda\,x, \lambda\,y)\,g(\lambda\,x, \lambda\,y) = \lambda^{r+2}\,f(x,y)\,\lambda^{3-r}\,g(x,y) = \lambda^5\,f(x,y)\,g(x,y) = \lambda^5\,h(x,y),$$

i.e., h is homogeneous of degree 5. Therefore, the Euler relation implies:

$$x\,h_x(x,y) + y\,h_y(x,y) = 5\,h(x,y).$$

Because

$$\varepsilon_{h,x}(x,y) = x\,\frac{h_x(x,y)}{h(x,y)}$$

we obtain:

$$\varepsilon_{h,x}(x,y) + y\,\frac{h_y(x,y)}{h(x,y)} = 5,$$

i.e.,

$$h(x,y) = \frac{y\,h_y(x,y)}{5 - \varepsilon_{h,x}(x,y)}.$$

We plug the expressions for $h_y(x,y)$ and $\varepsilon_{h,x}$ in this latter equation and obtain:

$$\begin{aligned}
h(x,y) &= \frac{y\left(-\frac{x^6}{y^2} + 3\,x^2\,y^2\right)}{5 - \frac{6\,x^6 + 2\,x^2\,y^4}{x^6 + x^2\,y^4}} \\
&= \frac{\left(-\frac{x^6}{y} + 3\,x^2\,y^3\right)\,(x^6 + x^2\,y^4)}{5\,x^6 + 5\,x^2\,y^4 - 6\,x^6 - 2\,x^2\,y^4} \\
&= \frac{(-x^6 + 3\,x^2\,y^4)\,(x^6 + x^2\,y^4)}{y\,(-x^6 + 3\,x^2\,y^4)} \\
&= \frac{x^6 + x^2\,y^4}{y}.
\end{aligned}$$

It follows that:

$$h(x,y) = \frac{x^6}{y} + x^2\,y^3.$$

Part II: Multiple-choice questions

Exercise 3

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☒
2.	☐	☐	☐	☒
3.	☐	☐	☒	☐
4.	☒	☐	☐	☐
5.	☐	☐	☐	☒
6.	☒	☐	☐	☐
7.	☒	☐	☐	☐
8.	☐	☐	☒	☐

1. Answer is (d). The following truth tables applies:

A	T	T	F	F
B	T	F	T	F
$A \vee B$	T	T	T	F
$A \wedge B$	T	F	F	F
$(A \vee B) \Rightarrow B$	T	F	T	T
$(A \vee B) \Leftrightarrow A$	T	T	F	T
$(A \wedge B) \Leftrightarrow A$	T	F	T	T
$(A \wedge B) \Rightarrow A$	**T**	**T**	**T**	**T**

Therefore, $(A \wedge B) \Rightarrow A$ is the only tautology.

2. Answer is (d). We have:

- (a) is wrong. Take as an example $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ and $f(x) = \frac{1}{x}$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and convergent and f is differentiable on $(0, \infty)$. However, the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n) = \frac{1}{\frac{1}{n}} = n$ is divergent.
- (b) is wrong. Take as an example $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ and $f(x) = x^2$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and convergent and f is differentiable on $\mathbb{R}$. However, the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n) = \left(\frac{1}{n}\right)^2 = \frac{1}{n^2}$ is monotone and convergent.
- (c) is wrong. Take as an example $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ and $f(x) = \sin(2\,\pi\,x)$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and convergent and f is differentiable on $\mathbb{R}$. However, the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n) = \sin\left(\frac{2\,\pi}{n}\right)$ is not monotone.

3. Answer is (c). (a) is clearly wrong, (b) is the definition of a decreasing function on $[0, 10]$, while (c) is the definition of an increasing function on $[0, 10]$. Finally, (d) is wrong as one can see from the following example: let $f(x) = x^2$ for $x \in (0, 10]$ and $f(0) = 100$. The function f is differential on $(0, 10)$ with $f'(x) = 2\,x > 0$ on $(0, 10)$. However, f is not increasing because $f(0) > f(x)$ for all $x \in (0, 10)$.

4. Answer is (a). Project I should be preferred with any strictly positive interest rate because it returns a total of 20,000 CHF (same as project II), but part of this amount sooner than project II does (i.e., already in 6 months). Formally, the present value of project I is $-100,000 + \frac{40,000}{(1+i)} + \frac{60,000}{(1+i)^2}$, while the present value of project II is $-110,000 + \frac{130,000}{(1+i)^2}$, where i is the semi-annual interest rate. Therefore, project I is preferred to project II when

$$\begin{aligned}
& -100,000 + \frac{40,000}{(1+i)} + \frac{80,000}{(1+i)^2} > -110,000 + \frac{130,000}{(1+i)^2} \\
\Leftrightarrow \quad & 10,000 + \frac{40,000}{(1+i)} - \frac{50,000}{(1+i)^2} > 0 \\
\Leftrightarrow \quad & 10,000\,(1+i)^2 + 40,000\,(1+i) - 50,000 > 0 \\
\Leftrightarrow \quad & (1+i)^2 + 4\,(1+i) - 5 > 0 \\
\Leftrightarrow \quad & 1 + 2\,i + i^2 + 4 + 4\,i - 5 > 0 \\
\Leftrightarrow \quad & i^2 + 2\,i > 0 \\
\Leftrightarrow \quad & i < -2 \text{ or } i > 0
\end{aligned}$$

Therefore, for any strictly positive interest rate, project I is preferred to project II.

5. Answer is (d). The only possible general statement with a strictly positive interest rate is that $C^D < C^I$, when $n^D \geq n^I$. This follows because payments at the beginning of the year generate higher interests compared to payments at the end of the year when the interest rate is strictly positive. Therefore (a), (b) and (c) are wrong.

6. Answer is (a). Clearly, f is continuous for $x \neq 0$. Moreover,

$$\begin{aligned}
\lim_{x\to 0} f(x) \quad & = \quad \lim_{x\to 0} \frac{(\sin(x))^2}{a\,x^2} \\
& \overset{\text{de l'Hôpital}}{=} \quad \lim_{x\to 0} \frac{2\,\sin(x)\,\cos(x)}{2\,a\,x} \\
& \overset{\text{de l'Hôpital}}{=} \quad \lim_{x\to 0} \frac{2\,(\cos^2(x) - \sin^2(x))}{2\,a} \\
& = \quad \frac{1}{a}.
\end{aligned}$$

It follows that f is continuous in $x = 0$ if and only if $\frac{1}{a} = a$, i.e., $a^2 = 1$. This latter equation have solution set $\{-1, 1\}$.

7. Answer is (a). The reminder term $R_4(x)$ at $x_0 = 0$ equals

$$R_4(x) = \frac{f^{(5)}(\xi)}{5!}\,x^5$$

for some ξ. Because $f^{(5)}(\xi) = \cdot 5!$ for all ξ, it follows that:

$$R_4(x) = 2\,x^5.$$

Therefore, $R_4(x) > 0$ for $x > 0$ and $R_4(x) = 0$ for $x = 0$.

8. Answer is (c). We have:

$$\varepsilon_f(x) = x\,\frac{f'(x)}{f(x)} = x\,\rho_f(x).$$

Therefore,

$$\rho_f(x) = \frac{\varepsilon_f(x)}{x} = \ln(x) + \frac{e^{3\,x}}{x}.$$

Exercise 4

	(a)	(b)	(c)	(d)
1.	☐	☐	☒	☐
2.	☐	☒	☐	☐
3.	☐	☐	☐	☒
4.	☐	☐	☐	☒
5.	☐	☒	☐	☐
6.	☐	☐	☐	☒
7.	☐	☐	☐	☒
8.	☐	☐	☒	☐

1. Answer is (c). First of all

$$\varepsilon_f(x) = x\,\frac{f'(x)}{f(x)} = x\,\frac{2\,x\,e^{-x} - x^2\,e^{-x}}{x^2\,e^{-x}} = 2 - x.$$

f is elastic (inelastic) at x_0 if and only if $|\varepsilon_f(x)| > 1$ (< 1). We have:

$$|\varepsilon_f(x)| < 1 \Leftrightarrow |2 - x| < 1 \Leftrightarrow (2 - x)^2 < 1 \Leftrightarrow (x - 3)\,(x - 1) < 0 \Leftrightarrow x \in (1, 3).$$

Therefore, (c) is correct.

2. Answer is (b). Because the square-root function is differentiable and strictly increasing, we consider the function $g(x) = x^2\,e^{x^2} + 1$ instead of the function f. Because g is also differentiable, a stationary point of g satisfies $g'(x) = 0$. We have:

$$g'(x) = 2\,x\,e^{x^2} + 2\,x^3\,e^{x^2} = 2\,(x + x^3)\,e^{x^2}.$$

Therefore, $g'(x) = 0$ if and only if $x = 0$, i.e., $x = 0$ is the only stationary point of g. Moreover,

$$g''(x) = 2\,(1 + 3\,x^2)\,e^{x^2} + (2\,x + 2\,x^3)\,e^{x^2}\,(2\,x) = 2\,e^{x^2}\,(1 + 5\,x^2 + 2\,x^4).$$

It follows that:

$$g''(0) = 2 > 0,$$

and $x = 0$ is a local minimum of g (and thus also of f).

3. Answer is (d). Because

$$P_5(x) = P_4(x) + \frac{f^{(5)}(0)}{5!}\,x^5,$$

we only have to determine $f^{(5)}$. We have:

$$\begin{aligned}
f'(x) &= \frac{-1}{(1+x)^2} \\
f''(x) &= \frac{2}{(1+x)^3} \\
f'''(x) &= \frac{-6}{(1+x)^4} \\
f^{(4)}(x) &= \frac{24}{(1+x)^5} \\
f^{(5)}(x) &= \frac{-120}{(1+x)^6}.
\end{aligned}$$

Therefore, $f^{(5)}(0) = -120$ and

$$P_5(x) = P_4(x) - x^5.$$

Depending on whether $x > 0$, $x < 0$ or $x = 0$, then $P_4(x) > P_5(x)$, $P_4(x) < P_5(x)$, or $P_4(x) = P_5(x)$.

4. Answer is (d). For function f,

$$(x, y) \in D_f \Leftrightarrow x^2 + y^2 - 25 \geq 0 \Leftrightarrow x^2 + y^2 \geq 5^2.$$

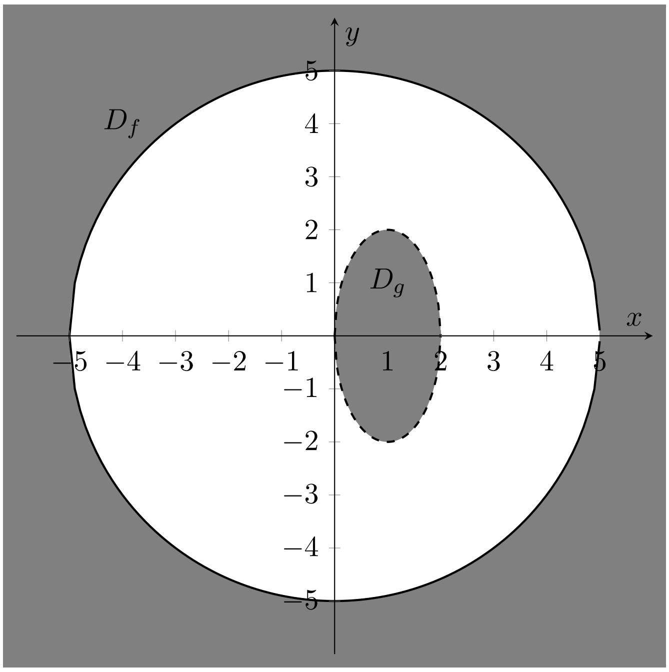
This corresponds to the area outside or on the circle with centre $(0, 0)$ and radius 5.

For function g,

$$(x, y) \in D_g \Leftrightarrow 8\,x - 4\,x^2 - y^2 > 0 \Leftrightarrow \frac{(x-1)^2}{1^2} + \frac{(y-0)^2}{2^2} < 1.$$

This corresponds to the area inside the ellipse with centre $(1, 0)$ and semi-axes $a = 1$ and $b = 2$.

Therefore, the domains of f and g do not intersect.



5. Answer is (b). Taylor theorem implies that:

$$R_4(x) = \frac{f^{(5)}(\xi)}{5!}\,x^5.$$

It follows that:

$$R_{f,4}(x) = 3\,x^5$$

and

$$R_{g,4}(x) = -2\,x^5.$$

Therefore,

$$R_{f,4}(x) > R_{g,4}(x)$$

for $x > 0$.

6. Answer is (d). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\, x, \lambda\, y) &= 8\left(\frac{1}{(\lambda\, x)^3} + \frac{1}{5\,(\lambda\, y)^3}\right)^{-\frac{1}{6}} + \sqrt[3]{3\,\lambda\, x} + \sqrt[6]{(\lambda\, y)^2} \\
&= 8\left[\frac{1}{\lambda^3}\left(\frac{1}{x^3} + \frac{1}{5\, y^3}\right)\right]^{-\frac{1}{6}} + \lambda^{\frac{1}{3}}\,\sqrt[3]{3\, x} + \lambda^{\frac{2}{6}}\,\sqrt[6]{y^2} \\
&= 8\,\lambda^{\frac{1}{2}}\left(\frac{1}{x^3} + \frac{1}{5\, y^3}\right)^{-\frac{1}{6}} + \lambda^{\frac{1}{3}}\,\sqrt[3]{3\, x} + \lambda^{\frac{1}{3}}\,\sqrt[6]{y^2} \\
&= \lambda^{\frac{1}{3}}\left\{8\,\lambda^{\frac{1}{2}-\frac{1}{3}}\left(\frac{1}{x^3} + \frac{1}{5\, y^3}\right)^{-\frac{1}{6}} + \sqrt[3]{3\, x} + \sqrt[6]{y^2}\right\} \\
&= \lambda^{\frac{1}{3}}\left\{8\,\lambda^{\frac{1}{6}}\left(\frac{1}{x^3} + \frac{1}{5\, y^3}\right)^{-\frac{1}{6}} + \sqrt[3]{3\, x} + \sqrt[6]{y^2}\right\}.
\end{aligned}$$

It follows that f is not homogeneous.

7. Answer is (d). For $\lambda > 0$ we have:

$$\begin{aligned}
g(\lambda\, x, \lambda\, y) &= f(\lambda\, a\, x, \lambda\, a^2\, y) \\
&= \frac{(\lambda\, a\, x)^3}{\lambda\, a^2\, y} + 1 + \sqrt{(\lambda\, a\, x)^4 + 2\,(\lambda\, a^2\, y)^4} \\
&= \lambda^2\,\frac{(a\, x)^3}{a^2\, y} + 1 + \lambda^2\,\sqrt{(a\, x)^4 + 2\,(a^2\, y)^4} \\
&= \lambda^2\left[\frac{(a\, x)^3}{a^2\, y} + 1 + \sqrt{(a\, x)^4 + 2\,(a^2\, y)^4}\right] + (1 - \lambda^2) \\
&= \lambda^2\, g(x, y) + (1 - \lambda^2).
\end{aligned}$$

Therefore, g is not homogeneous.

8. Answer is (c). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\, x, \lambda\, y) &= (\lambda\, x)^{a-1}\,(\lambda\, y)^{a+6} + \sqrt{(\lambda\, x)^2 + (\lambda\, y)^2} \\
&= \lambda^{a-1+a+6}\, x^{a-1}\, y^{a+6} + \lambda\,\sqrt{x^2 + y^2}.
\end{aligned}$$

Therefore, f is homogeneous if and only if

$$a - 1 + a + 6 = 1 \Leftrightarrow a = -2.$$