Mathematics A
Alternative Date Exam Autumn Semester 2017

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Part I: Open questions (50 points)

General instructions for open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain only a single version.

(vi) Temporary computations or sketches must be reported in separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (26 points)

(a1) (4 points)

Given is the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = \ln\left(\frac{1}{2 - e^{-3\sqrt{x+5}}}\right).$$

Determine the domain D_f and the range R_f of f.

Exercise 1

(a2) (3 points)

Given is the function

$$f\,:\,D_f \to \mathbb{R},\; x \mapsto y = \ln\left(\frac{1}{2 - e^{-3\sqrt{x+5}}}\right).$$

Is f strictly increasing on its domain (proof)?

Exercise 1

(a3) (3 points)

Given is the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = \ln\left(\frac{1}{2 - e^{-3\sqrt{x+5}}}\right).$$

Determine the inverse function f^{-1} of f.

Exercise 1

(b) (6 points)

You borrow $1,000,000$ CHF to finance a house. You enter a mortgage contract with an annual interest rate of 1% fixed for 10 years. You want to partially amortize your debt and thus pay an amount of 25,000 CHF at the end of each year for these first 10 years. After 10 years you need to renegotiate your mortgage contract. In the meantime interest rates increased and you finally agree on an annual interest rate of 2%. You pay C^I CHF at the end of each year for *additional* 15 years to bring your debt down to 200,000 CHF.

What is the payment C^I such that your plan is feasible?

Exercise 1

(b) (additional space for the solution)

Exercise 1

(c) (4 points)

Compute

$$\lim_{x \to 0} \frac{3^x - 2^x}{x^2 - x}.$$

Exercise 1

(d) (6 points)

The current data production is 2.5 Exabytes (10^{18} bytes) per day. Suppose that daily data production increases by 5% everyday. How many years starting from today does it take to reach a total data production of 45 Zetabytes (1 Zetabyte corresponds to 1000 Exabytes) assuming that 1 year always corresponds to 365 days?

Exercise 1

(d) (additional space for the solution)

Exercise 2 (24 points)

(a1) (5 points)

Let $a_k = \sin\left(\left(\frac{1}{4}\right)^k\right)$ for $k = 1, 2, \ldots$

Use the third order Taylor polynomial P_3 of the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = f(x) = \sin(x)$$

at $x_0 = 0$ to compute an approximation of $\sum_{k=1}^{\infty} a_k$.

Exercise 2

(a1) (additional space for the solution)

Exercise 2

(a2) (4 points)

Given is the function

$$f : D_f \to \mathbb{R}, \quad x \mapsto y = f(x) = \sin(x).$$

R_3 denotes the third order remainder term of f at $x_0 = 0$.

Show that

$$\sum_{k=1}^{\infty} R_3\left(\left(\frac{1}{4}\right)^k\right) \leq \frac{1}{6120}.$$

Exercise 2

(b) (4 points)

Given is the function

$$f(x, y) = \frac{\sqrt{16 - 16\,x^2 - y^2}}{(x + y)^2\,\sqrt{4 - x^2 - y^2}}.$$

Determine the domain D_f of f and represent it graphically.

Exercise 2

(b) (additional space for the solution)

Exercise 2

(c) (5 points)

Consider the production function

$$P(K, A) = \left(2\,K^{\alpha} + 4\,A^{\beta}\right)^2, \quad K > 0,\ A > 0, \quad \alpha > 0,\ \beta > 0.$$

Find α and β such that for the point $(K_0, A_0) = (2, 1)$ on the isoquant $P(K, A) = 144$ the substitution rate is $\frac{dA}{dK} = -1$.

Exercise 2

(c) (additional space for the solution)

Exercise 2

(d) (6 points)

The functions f and g are defined on $\mathbb{R}^2_{++}$ and have range $\mathbb{R}_{++}$. Moreover, the function f is homogeneous of degree $r+2$ and the function g is homogeneous of degree $3-r$, for $r \in (-2,3)$. The function h satisfies:

$$h(x,y) = f(x,y)\, g(x,y),$$

$$h_y(x,y) = -\frac{x^6}{y^2} + 3\,x^2\,y^2,$$

and

$$\varepsilon_{h,x}(x,y) = \frac{6\,x^6 + 2\,x^2\,y^4}{x^6 + x^2\,y^4}.$$

Determine h and simplify the terms of the function.

Exercise 2

(d) (additional space for the solution)

Part II: Multiple-choice question (50 points)

General instructions for multiple-choice questions:

(i) The solution must be reported in the multiple-choice solution form. Only the answers reported in the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility can be marked.

(iii) When two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions.

Exercise 3 (24 points)

Question 1 (4 points)

Which of the following propositions is a tautology?

(a) $(A \vee B) \Rightarrow B$.

(b) $(A \vee B) \Rightarrow A$.

(c) $(A \wedge B) \Leftrightarrow A$.

(d) $(A \wedge B) \Rightarrow A$.

Exercise 3

Question 2 (3 points)

Let f be a differentiable function. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence with $a_n \in D_f$ for all $n \in \mathbb{N}$ that is monotone and convergent. The sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n)$ for all $n \in \mathbb{N}$ is

(a) convergent.

(b) divergent.

(c) monotone.

(d) None of the above answers is correct.

Exercise 3

Question 3 (2 points)

A function $f : [0, 10] \to \mathbb{R}$ is increasing on $[0, 10]$ if and only if

(a) $f(10) \geq f(0)$.

(b) $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in [0, 10]$ with $x_1 < x_2$.

(c) $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in [0, 10]$ with $x_1 > x_2$.

(d) the first derivative $f'(x)$ exists for all $x \in (0, 10)$ and is positive.

Exercise 3

Question 4 (3 points)

An investor can choose between two projects. Project I requires an initial investment of 100, 000 CHF and returns 40, 000 CHF in 6 months and 80, 000 CHF in 1 year. Project II requires an initial investment of 110, 000 CHF and returns 130, 000 CHF in 1 year.

(a) Project I should be preferred to Project II if the interest rate is strictly positive.

(b) Project II should be preferred to Project I if the interest rate is strictly positive.

(c) Project I and Project II have the same net present value if the interest rate is strictly positive.

(d) Whether Project I should be preferred to Project II or Project II should be preferred to Project I depends on the value of the strictly positive interest rate.

Exercise 3

Question 5 (3 points)

A financial advisor suggests to his client two options to repay a mortgage: option 1 is to repay the mortgage with constant payments C^D at the beginning of each year for n^D years, while option 2 is to repay the same mortgage with constant payments C^I at the end of each year for n^I years.

Assuming that the interest rate is strictly positive, it follows that:

(a) $C^I = C^D$ when $n^I = n^D + 1$.

(b) $C^I = C^D$ when $n^I = n^D - 1$.

(c) $C^I = C^D$ when $n^I = n^D$.

(d) None of the answers above is correct.

Exercise 3

Question 6 (3 points)

We consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = \begin{cases} \frac{(\sin(x))^2}{a\,x^2} & \text{for } x \neq 0 \\ a & \text{for } x = 0 \end{cases}.$$

For which value of $a \in \mathbb{R} \setminus \{0\}$ is f overall continuous?

(a) For $a \in \{-1, 1\}$.

(b) For $a = 1$.

(c) For $a = -1$.

(d) For all $a \in \mathbb{R} \setminus \{0\}$.

Exercise 3

Question 7 (3 points)

Let $f(x) = 1 + 3\,x - 4\,x^4 + 2\,x^5$ and P_4 the forth order Taylor polynomial of f at $x_0 = 0$. Which of the following statements on the fourth order reminder term R_4 at $x_0 = 0$ is correct?

(a) $R_4(x) > 0$ for all $x > 0$.

(b) $R_4(x) < 0$ for all $x > 0$.

(c) $R_4(x) > 0$ for all $x \leq 0$.

(d) $R_4(x) < 0$ for all $x \leq 0$.

Exercise 3

Question 8 (3 points)

For a function f, the elasticity $\varepsilon_f(x)$ is:

$$\varepsilon_f(x) = x\,\ln(x) + e^{3\,x}.$$

For the growth rate $\rho_f(x)$ it follows that:

(a) $\rho_f(x) = x^2\,\ln(x) + x\,e^{3\,x}$.

(b) $\rho_f(x) = \ln(x) + e^{3\,x^2}$.

(c) $\rho_f(x) = \ln(x) + \frac{e^{3\,x}}{x}$.

(d) $\rho_f(x) = x\,\ln(x) + \frac{e^{3\,x}}{x}$.

Exercise 4 (26 points)

Question 1 (3 points)

Given is the function

$$f : \mathbb{R}_{++} \to \mathbb{R}, \quad x \mapsto f(x) = x^2 e^{-x}.$$

(a) f is elastic for $x > 0$.

(b) f is elastic for $x < 2$ and inelastic for $x > 2$.

(c) f is inelastic for $x \in (1, 3)$ and elastic for $x < 1$ and $x > 3$.

(d) f is inelastic for $x > 0$.

Exercise 4

Question 2 (4 points)

Given is the function

$$f : \mathbb{R} \to \mathbb{R}_{++}, \quad x \mapsto f(x) = \sqrt{x^2 \, e^{x^2} + 1}.$$

(a) f has a local maximum at $x_0 = 0$.

(b) f has a local minimum at $x_0 = 0$.

(c) f has an inflection point at $x_0 = 0$.

(d) f has no stationary points.

Exercise 4

Question 3 (4 points)

Consider the function

$$f : \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ x \mapsto f(x) = \frac{-1}{1+x}$$

and let P_4 and P_5 be the fourth and fifth order Taylor polynomials of f at $x_0 = 0$. It follows that:

(a) $P_4(x) > P_5(x)$ for all $x \in D_f \setminus \{x_0\}$.

(b) $P_4(x) < P_5(x)$ for all $x \in D_f \setminus \{x_0\}$.

(c) $P_4(x) = P_5(x)$ for all $x \in D_f$.

(d) $P_4(x) > P_5(x)$, $P_4(x) < P_5(x)$ or $P_4(x) = P_5(x)$ are all possible for some $x \in D_f$.

Exercise 4

Question 4 (4 points)

Given are the functions

$$f(x,y) = \sqrt{x^2 + y^2 - 25}$$

and

$$g(x,y) = \ln\left(8\,x - 4\,x^2 - y^2\right),$$

and D_f and D_g are the corresponding domains.

We have:

(a) $D_f \subseteq D_g$.

(b) $D_g \subseteq D_f$.

(c) $D_f = D_g$.

(d) $D_f \cap D_g = \emptyset$.

Exercise 4

Question 5 (3 points)

Let $f(x) = 1 - x + 3\,x^2 - 4\,x^3 + 5\,x^4 + 3\,x^5$ and $g(x) = 1 + 4\,x^2 - 6\,x^3 + 4\,x^4 - 2\,x^5$ and let $P_{f,4}$ and $P_{g,4}$ their respective fourth order Taylor polynomials at $x_0 = 0$.

Which of the following statements concerning the fourth order reminder terms $R_{f,4}$ and $R_{g,4}$ of f and g, respectively, at $x_0 = 0$ is correct?

(a) $R_{f,4}(x) < R_{g,4}(x)$ for all $x > 0$.

(b) $R_{f,4}(x) > R_{g,4}(x)$ for all $x > 0$.

(c) $R_{f,4}(x) = R_{g,4}(x)$ for all $x > 0$.

(d) None of the above answers is correct.

Exercise 4

Question 6 (2 points)

Given is the function

$$f(x,y) = 8\left(\frac{1}{x^3} + \frac{1}{5\,y^3}\right)^{-\frac{1}{6}} + \sqrt[3]{3\,x} + \sqrt[6]{y^2} \quad (x > 0,\, y > 0).$$

(a) f is linear homogeneous.

(b) f is homogeneous of degree -0.5.

(c) f is homogeneous of degree 0.5.

(d) f is not homogeneous.

Exercise 4

Question 7 (3 points)

Given is the function

$$f(x,y) = \frac{x^3}{y} + 1 + \sqrt{x^4 + 2\,y^4} \quad (x > 0,\, y > 0)$$

and

$$g(x,y) = f(a\,x, a^2\,y),$$

where $a > 0$.

(a) g is linear homogeneous.

(b) g is homogeneous of degree a.

(c) g is homogeneous of degree $2\,a$.

(d) g is not homogeneous.

Exercise 4

Question 8 (3 points)

For which value of $a \in \mathbb{R}$ is the function

$$f(x,y) = x^{a-1}\,y^{a+6} + \sqrt{x^2+y^2} \quad (x > 0,\, y > 0)$$

homogeneous?

(a) f is homogeneous for $a = 0$.

(b) f is homogeneous for $a = 1$.

(c) f is homogeneous for $a = -2$.

(d) f is for no $a \in \mathbb{R}$ homogeneous.

Exams Assessment Level - Autumn Term 2017

1,202 Mathematics A

Multiple-choice answer sheet, Page 1 of 2

Exercise 3 (24 points)

Question 1: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Exams Assessment Level - Autumn Term 2017

1,202 Mathematics A

Multiple-choice answer sheet, Page 2 of 2

Exercise 4 (26 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐