Mathematics B
Master Solution Alternative Date Exam
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Prof. Dr. Enrico De Giorgi*

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*Faculty of Mathematics and Statistics, University St. Gallen, Bodanstrasse 6, 9000 St. Gallen, Switzerland, email: enrico.degiorgi@unisg.ch.

Part I: Open Questions

Exercise 1

(a) (7 points)
The necessary conditions for maxima, minima or saddle points $(x^\star, y^\star)$ of f are

$$\begin{cases} f_x(x^\star, y^\star) = 0 \\ f_y(x^\star, y^\star) = 0 \end{cases}.$$

We thus compute the first order partial derivatives of f and we obtain

$$\begin{aligned} f_x(x, y) &= 2\,a\,y + 4\,x^3, \\ f_y(x, y) &= 2\,a\,y + 2\,a\,x. \end{aligned}$$

It follows that:

$$\begin{cases} f_x(x^\star, y^\star) = 0 \\ f_y(x^\star, y^\star) = 0 \end{cases} \Leftrightarrow \begin{cases} 2\,a\,y^\star + 4\,(x^\star)^3 = 0 \\ 2\,a\,y^\star + 2\,a\,x^\star = 0 \end{cases}.$$

From $2\,a\,y^\star + 2\,a\,x^\star = 0$ we obtain

$$x^\star = -y^\star.$$

We plug this latter result into $2\,a\,y^\star + 4\,(x^\star)^3 = 0$ and we find

$$\begin{aligned} 4\,(x^\star)^3 = 2\,a\,x^\star \quad &\Leftrightarrow \quad 2\,x^\star\,\left(2\,(x^\star)^2 - a\right) = 0 \\ &\Leftrightarrow \quad x^\star \in \left\{-\sqrt{\frac{a}{2}}, 0, \sqrt{\frac{a}{2}}\right\}. \end{aligned}$$

Using that $y^\star = -x^\star$, we obtain three stationary points for f:

$$P_1 = (0, 0)\,,$$

$$P_2 = \left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right),$$

and

$$P_3 = \left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right).$$

Next, we verify the sufficient conditions: if $(x^\star, y^\star)$ satisfies the necessary conditions, then the following holds:

$$\begin{cases} f_{xx}(x^\star, y^\star) > 0 \\ f_{yy}(x^\star, y^\star) > 0 \\ f_{xx}(x^\star, y^\star)\, f_{yy}(x^\star, y^\star) - (f_{xy}(x^\star, y^\star))^2 > 0 \end{cases} \Rightarrow (x^\star, y^\star) \text{ is a minimum},$$

$$\begin{cases} f_{xx}(x^\star, y^\star) < 0 \\ f_{yy}(x^\star, y^\star) < 0 \\ f_{xx}(x^\star, y^\star)\, f_{yy}(x^\star, y^\star) - (f_{xy}(x^\star, y^\star))^2 > 0 \end{cases} \Rightarrow (x^\star, y^\star) \text{ is a maximum},$$

and

$$f_{xx}(x^\star, y^\star)\, f_{yy}(x^\star, y^\star) - (f_{xy}(x^\star, y^\star))^2 < 0 \Rightarrow (x^\star, y^\star) \text{ is a saddle point.}$$

For the sufficient conditions we need the second order partial derivatives of f. We have:

$$\begin{aligned} f_{xx}(x,y) &= 12\,x^2, \\ f_{yy}(x,y) &= 2\,a, \\ f_{xy}(x,y) &= 2\,a. \end{aligned}$$

It follows that:

$$f_{xx}(x^\star,y^\star)\,f_{yy}(x^\star,y^\star) - (f_{xy}(x^\star,y^\star))^2 = 24\,(x^\star)^2\,a - 4\,a^2.$$

For $P_1 = (0,0)$ we have:

$$f_{xx}(0,0)\,f_{yy}(0,0) - (f_{xy}(0,0))^2 = -4\,a^2 < 0.$$

Therefore, $P_1 = (0,0)$ is a saddle point.

For $P_2 = \left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$ we have (note that $a \in \mathbb{R}_{++}$):

$$\left\{ \begin{array}{c} f_{xx}\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right) = 6\,a > 0 \\ f_{yy}\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right) = 2\,a > 0 \\ f_{xx}\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)\, f_{yy}\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right) - \left(f_{xy}\left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)\right)^2 = 8\,a^2 > 0 \end{array} \right. .$$

Therefore, $P_2 = \left(\sqrt{\frac{a}{2}}, -\sqrt{\frac{a}{2}}\right)$ is a local minimum.

For $P_3 = \left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)$ we have (note that $a \in \mathbb{R}_{++}$):

$$\left\{ \begin{array}{c} f_{xx}\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right) = 6\,a > 0 \\ f_{yy}\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right) = 2\,a > 0 \\ f_{xx}\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)\, f_{yy}\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right) - \left(f_{xy}\left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)\right)^2 = 8\,a^2 > 0 \end{array} \right. .$$

Therefore, $P_3 = \left(-\sqrt{\frac{a}{2}}, \sqrt{\frac{a}{2}}\right)$ is a local minimum.

(b) (7 points)
We apply the Lagrange method. First of all we define the Lagrange function:

$$\begin{aligned} F(x,y,\lambda) &= f(x,y) + \lambda\,\varphi(x,y) \\ &= 2\,x^2 + y^2 + \lambda\,(x^2 - a\,x\,y + b\,y^2 + 15). \end{aligned}$$

The necessary conditions for constrained extreme points of f, given the constraint $\varphi(x,y) = 0$, are the so-called Lagrange conditions:

$$\begin{aligned} F_x(x,y,\lambda) = 0 \;&\Rightarrow\; 4\,x + \lambda\,(2\,x - a\,y) = 0, && \text{(I)} \\ F_y(x,y,\lambda) = 0 \;&\Rightarrow\; 2\,y + \lambda\,(-a\,x + 2\,b\,y) = 0, && \text{(II)} \\ F_\lambda(x,y,\lambda) = 0 \;&\Rightarrow\; x^2 - a\,x\,y + b\,y^2 + 15 = 0. && \text{(III)} \end{aligned}$$

Therefore, in order that $(x,y) = (1,2)$ is a constrained extreme point of f, we must have:

$$\begin{aligned} F_x(1,2,\lambda) = 0 \;&\Rightarrow\; 4 + \lambda\,(2 - 2\,a) = 0, && \text{(IV)} \\ F_y(1,2,\lambda) = 0 \;&\Rightarrow\; 4 + \lambda\,(-a + 4\,b) = 0, && \text{(V)} \\ F_\lambda(1,2,\lambda) = 0 \;&\Rightarrow\; 1 - 2\,a + 4\,b + 15 = 0. && \text{(VI)} \end{aligned}$$

From (IV) we obtain:

$$\lambda = \frac{4}{2\,a - 2}. \qquad \text{(VII)}$$

From (V) we obtain:

$$\lambda = \frac{4}{a - 4\,b}. \qquad \text{(VIII)}$$

Equations (VII) and (VIII) imply that:

$$\frac{4}{2\,a - 2} = \lambda = \frac{4}{a - 4\,b} \quad \Leftrightarrow \quad 2\,a - 2 = a - 4\,b \quad \Leftrightarrow \quad a = 2 - 4\,b.$$

We plug this result into (VI) and we have:

$$1 - 2\,(2 - 4\,b) + 4\,b + 15 = 0 \quad \Leftrightarrow \quad -12\,b = 12 \quad \Leftrightarrow \quad b = -1 \quad \Rightarrow \quad a = 6.$$

(c) (5 points)
First of all, applying the property $\ln(x^r) = r\,\ln(x)$ of the logarithm, we get:

$$\int_1^e x^2\,\ln(\sqrt{x})\,dx = \frac{1}{2}\int_1^e x^2\,\ln(x)\,dx.$$

We compute the indefinite integral $\int x^2\,\ln(x)\,dx$ using integration by parts. Let

$$u'(x) = x^2 \qquad \text{and} \qquad v(x) = \ln(x).$$

Then,

$$u = \frac{1}{3}\,x^3 \qquad \text{and} \qquad v'(x) = \frac{1}{x}.$$

It follows that:

$$\begin{aligned}
\int \underbrace{x^2}_{=u'(x)}\,\underbrace{\ln(x)}_{=v(x)}\,dx &= \underbrace{\frac{1}{3}\,x^3}_{=u(x)}\,\underbrace{\ln(x)}_{=v(x)} - \int \underbrace{\frac{1}{3}\,x^3}_{=u(x)}\,\underbrace{\frac{1}{x}}_{=v'(x)}\,dx \\
&= \frac{1}{3}\,x^3\,\ln(x) - \frac{1}{3}\int x^2\,dx \\
&= \frac{1}{3}\,x^3\,\ln(x) - \frac{1}{9}\,x^3 + C, \qquad C \in \mathbb{R} \\
&= \frac{1}{3}\,x^3\left(\ln(x) - \frac{1}{3}\right) + C.
\end{aligned}$$

We now compute the definite integral:

$$\begin{aligned}
\int_1^e x^2\,\ln(\sqrt{x})\,dx &= \frac{1}{2}\left[\frac{1}{3}\,x^3\left(\ln(x) - \frac{1}{3}\right)\right]_1^e \\
&= \frac{1}{6}\,e^3\left(1 - \frac{1}{3}\right) - \frac{1}{6}\left(0 - \frac{1}{3}\right) \\
&= \frac{2\,e^3 + 1}{18} \approx 2.287.
\end{aligned}$$

(d) (6 points)

We solve the indefinite integral $\int \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx$ using the substitution rule. Let $u = x\,e^x$, then:

$$\frac{du}{dx} = e^x + x\,e^x$$

and

$$(e^x + x\,e^x)\,dx = du.$$

We obtain:

$$\int \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx = \int \frac{1}{u^3}\,du = -\frac{1}{2\,u^2} + C = -\frac{1}{2\,(x\,e^x)^2} + C.$$

We now compute the improper integral:

$$\begin{aligned}
\int_1^\infty \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx &= \lim_{b\to\infty} \int_1^b \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{2\,(x\,e^x)^2}\right]_1^b \\
&= \lim_{b\to\infty} \left[-\frac{1}{2\,(b\,e^b)^2} + \frac{1}{2\,e^2}\right] \\
&= \underbrace{\lim_{b\to\infty} \left[-\frac{1}{2\,(b\,e^b)^2}\right]}_{\longrightarrow 0} + \frac{1}{2\,e^2} \\
&= \frac{1}{2\,e^2} \approx 0.0677
\end{aligned}$$

Exercise 2

(a) (4 points)

We have to prove the following:

$$C^T = C.$$

We obtain:

$$\begin{aligned} C^T &= \left[B^{-1} A\, A^{-T} (B^T)^{-1}\right]^T \\ &= \left[(B^T)^{-1}\right]^T (A^T)^T A^T (B^{-1})^T \\ &= \left[(B^T)^T\right]^{-1} A\, A^T (B^T)^{-1} \\ &= B^{-1} A\, A^T (B^T)^{-1} \\ &= C. \end{aligned}$$

(b) (4 points)

The direction of the steepest descent of f at the point $(x_0, y_0) = (-1, \pi)$ is given by the vector

$$\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

if and only if

$$\lambda\, \mathbf{b} = \mathbf{grad} f(-1, \pi)$$

for some $\lambda < 0$.

We have:

$$\mathbf{grad} f(x, y) = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} = \begin{pmatrix} \sin(y) + \frac{a}{x+2} + \frac{y}{\pi} \\ x\, \cos(y) + \frac{x}{\pi} \end{pmatrix}.$$

It follows that:

$$\mathbf{grad} f(-1, \pi) = \begin{pmatrix} a+1 \\ 1 - \frac{1}{\pi} \end{pmatrix}.$$

Therefore, we have:

$$\begin{aligned} & \lambda\, \mathbf{b} = \mathbf{grad} f(-1, \pi) \\ \Leftrightarrow \quad & \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} a+1 \\ 1 - \frac{1}{\pi} \end{pmatrix}. \end{aligned}$$

It follows (2^{nd} component) that $\lambda = \frac{1}{\pi} - 1 < 0$ and

$$a + 1 = 2\left(\frac{1}{\pi} - 1\right) \quad \Leftrightarrow \quad a = \frac{2}{\pi} - 3 \approx -2.3634.$$

(c) (3 points)
Since we have three linearly independent equations, it holds for the dimension of V that

$$\dim(V) = 5 - 3 = 2.$$

In particular, x_4 and one more variable are free variables. We choose

$$x_1 = t, \quad x_4 = s.$$

We now solve for the other variables x_2, x_3, x_5:

$$\begin{aligned}
x_1 + x_2 = 0 &\quad\Rightarrow\quad x_2 = -x_1 = -t, \\
x_1 + 5\,x_2 - 2\,x_3 = 0 &\quad\Rightarrow\quad 2\,x_3 = x_1 + 5\,x_2 = -4\,t \quad\Rightarrow\quad x_3 = -2\,t, \\
x_1 - x_3 + x_5 = 0 &\quad\Rightarrow\quad x_5 = -x_1 + x_3 = -3\,t.
\end{aligned}$$

Therefore, we get the following basis of $V \subset \mathbb{R}^5$:

$$\mathcal{B}(V) = \left\{ \mathbf{b_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \\ -3 \end{pmatrix}, \mathbf{b_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(d) (6 points)
λ is an eigenvalue of M if and only if $\det(M - \lambda\, I) = 0$. We have:

$$\begin{aligned}
\det(M - \lambda\, I) &= \left| \begin{pmatrix} -\lambda & 3 & 0 \\ 1 & 2-\lambda & 0 \\ -5\,a & 5 & 4-\lambda \end{pmatrix} \right| \\
&= -\lambda\,(2-\lambda)\,(4-\lambda) - 3\,(4-\lambda) \\
&= (4-\lambda)\,(\lambda^2 - 2\lambda - 3) \\
&= (4-\lambda)\,(\lambda - 3)\,(\lambda + 1)
\end{aligned}$$

and thus

$$\det(M - \lambda\, I) = 0 \quad\Leftrightarrow\quad \lambda \in \{-1, 3, 4\}.$$

In particular, $\lambda = 3$ is an eigenvalue of M.

We now compute the eigenvectors corresponding to the eigenvalue $\lambda = 3$. The vector $\mathbf{x} = (x_1, x_2, x_3)^T$ is an eigenvector of M associated with the eigenvalue λ if and only if $(M - \lambda\, I)\,\mathbf{x} = \mathbf{0}$.

For $\lambda = 3$ we have (applying the Gauss method):

$$
\begin{array}{rcl}
(M - 3\,I, \mathbf{0}) & = & \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -5\,a & 5 & 1 & 0 \end{array}\right) \end{array} \begin{array}{l} \\ :(-3) \\ \\ \\ \end{array} \\
& \to & \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -5\,a & 5 & 1 & 0 \end{array}\right) \begin{array}{l} \\ +(I) \\ +5\,a\,(I) \end{array} \\
& \to & \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 - 5\,a & 1 & 0 \end{array}\right) ,
\end{array}
$$

i.e.:

$$
\begin{aligned}
x_1 \quad -x_2 \quad &= 0, \\
(5 - 5\,a)\,x_2 + x_3 &= 0.
\end{aligned}
$$

We choose $x_2 = t$ and obtain:

$$
\begin{aligned}
& x_1 = x_2 = t, \\
& x_2 = t, \\
& x_3 = (5\,a - 5)\,x_2 = (5\,a - 5)\,t.
\end{aligned}
$$

Therefore, the eigenspace, i.e. the set of all eigenvalues plus the vector $\mathbf{v} = \mathbf{0}$, is given by

$$
V_{(\lambda=3)} = \left\{ \mathbf{x} \in \mathbb{R}^3 \,:\, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 5\,a - 5 \end{pmatrix}; \quad t \in \mathbb{R} \right\}.
$$

(e) (6 points)

We apply the Gauss method:

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 4 & 0 & 4 & 8 \\ 12 & 16 & 8 & 36+4\,r \\ 0 & 1 & -2 & s \\ 3 & 2 & 3 & 9-2\,r+s \\ 1 & 1 & 1 & 3 \end{array}\right) \begin{array}{l} :4 \\ \\ \\ \\ \\ \end{array} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 12 & 16 & 8 & 36+4\,r \\ 0 & 1 & -2 & s \\ 3 & 2 & 3 & 9-2\,r+s \\ 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \\ -12\,(I) \\ \\ -3\,(I) \\ -I \end{array} \end{array}$$

$$\rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 16 & -4 & 12+4\,r \\ 0 & 1 & -2 & s \\ 0 & 2 & 0 & 3-2\,r+s \\ 0 & 1 & 0 & 1 \end{array}\right) \end{array} \overset{(II)\leftrightarrow(V)}{\rightarrow} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & s \\ 0 & 2 & 0 & 3-2\,r+s \\ 0 & 16 & -4 & 12+4\,r \end{array}\right) \begin{array}{l} \\ \\ -(II) \\ -2\,(II) \\ -16\,(II) \end{array} \end{array}$$

$$\rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & s-1 \\ 0 & 0 & 0 & 1-2\,r+s \\ 0 & 0 & -4 & -4+4\,r \end{array}\right) \begin{array}{l} \\ \\ :(-2) \\ \\ \\ \end{array} \end{array} \rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2}-\frac{1}{2}s \\ 0 & 0 & 0 & 1-2\,r+s \\ 0 & 0 & -4 & -4+4\,r \end{array}\right) \begin{array}{l} -(III) \\ \\ \\ \\ -4\,(III) \end{array} \end{array}$$

$$\rightarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2}+\frac{1}{2}s \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2}-\frac{1}{2}s \\ 0 & 0 & 0 & 1-2\,r+s \\ 0 & 0 & 0 & -2+4\,r-2\,s \end{array}\right) \begin{array}{l} -(III) \\ \\ \\ \\ -4\,(III) \end{array} \end{array}$$

Therefore, the system of equation is solvable if and only if:

$$1-2\,r+s=0 \ \wedge \ -2+4\,r-2\,s=0 \quad \Rightarrow \quad s=2\,r-1.$$

Hence an unique solution exists for $s=2\,r-1$, $r\in\mathbb{R}$ and corresponds to:

$$\mathbf{x}=\begin{pmatrix} \frac{3}{2}+\frac{1}{2}\,s \\ 1 \\ \frac{1}{2}-\frac{1}{2}\,s \end{pmatrix}=\begin{pmatrix} r+1 \\ 1 \\ 1-r \end{pmatrix}.$$

Part II: Multiple-choice Questions

Exercise 3

	(a)	(b)	(c)	(d)
Question 1	☐	☐	☒	☐
Question 2	☐	☐	☐	☒
Question 3	☐	☒	☐	☐
Question 4	☐	☐	☐	☒
Question 5	☒	☐	☐	☐
Question 6	☐	☒	☐	☐
Question 7	☒	☐	☐	☐
Question 8	☐	☐	☐	☒

Q1. (c). The constraint $\varphi(x,y) = \frac{x^2}{4} + \frac{y^2}{25} - 1 = 0$ corresponds to an ellipse with centre $(0,0)$ and semi-axes 2 and 5. The function $f(x,y) = -x$ takes its minimum if x gets maximal. The point on the ellipse with the largest x-value is $P = (2,0)$. Note that the point $P = (3,1)$ does not fulfill the constraint.

Q2. (d). Since $f(2) = -1 < 0$ for all $a \in \mathbb{R}$, f cannot be a density function.

Q3. (b). Because $\mathrm{rg}(A) \leq \min\{6,4\} = 4$ and $\mathrm{rg}(A) = \mathrm{rg}(A^T)$, only answer (b) can be true.

Q4. (d). There is no general relationship for the determinant of a sum of matrices.

Q5. (a). Because $\det([\mathbf{a},\mathbf{b},\mathbf{c}]) = 0$, the system of vectors $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is linearly dependent. In particular, we have $\mathbf{a}+\mathbf{b} = \mathbf{c}$. Therefore, $\mathbf{d}$ is a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ if and only if it is a linear combination of $\mathbf{a}$ and $\mathbf{b}$. This is true if and only if the system $\{\mathbf{a},\mathbf{b},\mathbf{d}\}$ is linearly dependent, that is, if and only if

$$\det([\mathbf{a},\mathbf{b},\mathbf{d}]) = \left|\begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & 4 \\ 3 & 1 & t \end{pmatrix}\right| = 3\,t - 18 = 0 \quad \Leftrightarrow \quad t = 6.$$

We have:

$$\mathbf{a} + 3\,\mathbf{b} = \mathbf{d}.$$

Q6. (b). Because $\mathrm{rg}(A) \leq \mathrm{rg}(A,\mathbf{b}) = 4$, the system has an unique solution if and only if $\mathrm{rg}(A) = 4$ and no solution for $\mathrm{rg}(A) < 4$.

Q7. (a). Let $f(x) = \frac{x}{5} - \frac{2}{25}\ln(5\,x+2) + C$, for some $C \in \mathbb{R}$. We have:

$$f'(x) = \frac{1}{5} - \frac{2}{25}\,\frac{5}{5\,x+2} = \frac{(5\,x+2)-2}{5\,(5\,x+2)} = \frac{x}{5\,x+2}.$$

Therefore,

$$\int \frac{x}{5\,x+2}\,dx = \frac{x}{5} - \frac{2}{25}\,\ln(5\,x+2) + C.$$

It can be easily shown that the functions in answers (b) and (c) are *no* antiderivatives of $\frac{x}{5\,x+2}$.

Q8. (d). We have

$$A\,\mathbf{b} = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} -20 \\ -10 \end{pmatrix} = 5\,\mathbf{b}.$$

Hence $\mathbf{b}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 5$ and it holds that

$$A^n\,\mathbf{b} = 5^n\,\mathbf{b} \text{ for all } n \in \mathbb{N}.$$

Therefore, (a) - (c) are wrong.

Exercise 4

	(a)	(b)	(c)	(d)
Question 1	☐	☒	☐	☐
Question 2	☐	☐	☒	☐
Question 3	☐	☒	☐	☐
Question 4	☐	☐	☐	☒
Question 5	☐	☐	☒	☐
Question 6	☐	☒	☐	☐
Question 7	☐	☐	☐	☒
Question 8	☐	☒	☐	☐

Q1. (b). We have

$$\begin{aligned}\int_{-\infty}^{1} e^{2\,x+1}\,dx &= \lim_{a\to-\infty}\left[\frac{1}{2}\,e^{2\,x+1}\right]_a^1\\ &= \lim_{a\to-\infty}\left(\frac{e^3}{2}-\frac{e^{2\,a+1}}{2}\right)\\ &= \frac{e^3}{2}-\underbrace{\lim_{a\to-\infty}\frac{e^{2\,a+1}}{2}}_{\longrightarrow 0}\\ &= \frac{e^3}{2}.\end{aligned}$$

Q2. (c). We have:

$$\mathbf{grad} f(x,y) = \begin{pmatrix} 2\,x \\ -\frac{1}{y}\end{pmatrix}.$$

$\mathbf{grad} f(t,t)$ is orthogonal to $\mathbf{b}$ if and only if

$$\begin{aligned} &\mathbf{grad} f(t,t)\cdot\mathbf{b} = 0\\ \Leftrightarrow\quad & 2\,t-\frac{1}{t}=0\\ \Leftrightarrow\quad & t^2=\frac{1}{2}\\ \overset{t>0}{\Rightarrow}\quad & t=\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}.\end{aligned}$$

Q3. (b). We apply the Gauss method:

$$A \quad = \quad \begin{pmatrix} 1 & 1 & -3 & 5 & 6\\ -2 & -1 & 2 & 4 & 9\\ 9 & 5 & -10 & 0 & 1\\ 2 & 0 & 3 & -7 & 1\end{pmatrix}\begin{array}{l} \\ +2\,(I)\\ -9\,(I)\\ -2\,(I)\end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & -3 & 5 & 6 \\ 0 & 1 & -4 & 14 & 21 \\ 0 & -4 & 17 & -45 & -53 \\ 0 & -2 & 9 & -17 & -11 \end{pmatrix} \begin{array}{l} -(II) \\ \\ +4\,(II) \\ +2\,(II) \end{array}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & -9 & -15 \\ 0 & 1 & -4 & 14 & 21 \\ 0 & 0 & 1 & 11 & 31 \\ 0 & 0 & 1 & 11 & 31 \end{pmatrix} \begin{array}{l} -(III) \\ +4\,(III) \\ \\ -(III) \end{array}$$

$$A^\star = \begin{pmatrix} 1 & 0 & 0 & -20 & -46 \\ 0 & 1 & 0 & 58 & 145 \\ 0 & 0 & 1 & 11 & 31 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\mathrm{rg}(A^\star) = 3$ and thus $\mathrm{rg}(A) = \mathrm{rg}(A^\star) = 3$.

Q4. (d). We apply the Gauss method:

$$(A|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ :(-1) \\ +(I) \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right) \begin{array}{l} -(II) \\ \\ -(II) \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \begin{array}{l} \\ -(III) \\ \\ \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) = (I|A^{-1})$$

$$\Rightarrow \quad A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Q5. (c).

$$A\,\mathbf{b} = \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -8 \\ 6 \end{pmatrix} = \begin{pmatrix} 16 \\ -12 \end{pmatrix} = -2 \begin{pmatrix} -8 \\ 6 \end{pmatrix} = -2\,\mathbf{b}.$$

Q6. (b). We plug $y_0 = 2$ and $y_1 = \frac{5}{3}$ into the difference equation (for $k = 1$) and obtain:

$$(1-a)\,y_1 + a\,y_0 - 1 = 0$$

$$\begin{aligned}
&\Leftrightarrow \quad (1-a)\,\frac{5}{3} + 2\,a - 1 = 0 \\
&\Leftrightarrow \quad \frac{5}{3} - \frac{5}{3}\,a + 2\,a - 1 = 0 \\
&\Leftrightarrow \quad \frac{1}{3}\,a = -\frac{2}{3} \\
&\Leftrightarrow \quad a = -2.
\end{aligned}$$

Q7. (d). We obtain the normal form of the difference equation:

$$\begin{aligned}
&3\,(y_k - 2\,y_{k+1}) = 6 - 7\,y_k \\
&\quad \Leftrightarrow \quad 3\,y_k - 6\,y_{k+1} = 6 - 7\,y_k \\
&\quad \Leftrightarrow \quad -6\,y_{k+1} = -10\,y_k + 6 \\
&\quad \Leftrightarrow \quad y_{k+1} = \frac{5}{3}\,y_k - 1.
\end{aligned}$$

It follows that $A = \frac{5}{3}$ and $B = -1$. Because $A > 0$ and $|A| > 1$, the general solution of the difference equation is monotone and divergent.

Q8. (b). The normal form of the difference equation is

$$y_{k+1} = \frac{1+c}{c-2}\,y_k + \frac{2}{2-c}.$$

The general solution of the difference equation is divergent and oscillating if and only if

$$A = \frac{1+c}{c-2} \leq -1.$$

Case 1: $c > 2$

$$\begin{aligned}
1 + c &\leq -c + 2 \\
2\,c &\leq 1 \\
c &\leq \frac{1}{2}.
\end{aligned}$$

Impossible.

Case 2: $c < 2$

$$\begin{aligned}
1 + c &\geq -c + 2 \\
2\,c &\geq 1 \\
c &\geq \frac{1}{2}.
\end{aligned}$$

Hence, $c \in \left[\frac{1}{2}, 2\right)$.