Mathematics B
Alternative Date Exam Spring Semester 2017

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Part I: Open Questions (50 points)

General instructions for the open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain a single version.

(vi) Temporary computations or sketches must be reported in separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (25 points)

(a) (7 points)

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables defined by

$$f(x, y) = a\,y^2 + 2axy + x^4 \quad (a \in \mathbb{R}_{++}).$$

Examine the function f for *stationary points*, i.e., maxima, minima, and saddle points.

Exercise 1

(a) (additional space for your solution)

Exercise 1

(b) (7 points)

The function

$$f(x, y) = 2x^2 + y^2$$

has to be optimised under the constraint

$$\varphi(x, y) = x^2 - axy + by^2 + 15 = 0.$$

Determine the parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $(1, 2)$ is a possible extreme point.

Remark:
It is not required to verify if $(1, 2)$ is indeed an extreme point, and which type (maximum or minimum) of extreme point is present.

Exercise 1

(b) (additional space for your solution)

Exercise 1

(c) (5 points)

Compute

$$\int_1^e x^2 \ln(\sqrt{x})\, dx.$$

Exercise 1

(c) (additional space for your solution)

Exercise 1

(d) (6 points)

Compute

$$\int_1^\infty \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx.$$

Exercise 1

(d) (additional space for your solution)

Exercise 2 (25 points)

(a) (4 points)

Let the square $n \times n$ matrices A and B be regular.

Prove the following statement: The matrix $C = B^{-1}\, A\, A^T\, (B^T)^{-1}$ is symmetric.

Exercise 2

(b) (4 points)

Given is the function

$$f(x, y) = x \sin(y) + a \ln(x + 2) + \frac{x\, y}{\pi}, \quad \text{where } a \in \mathbb{R}.$$

Compute the gradient of f at $(x_0, y_0) = (-1, \pi)$.

How do you have to choose the parameter $a \in \mathbb{R}$, such that the direction of the steepest *descent* of the function f at the point $(x_0, y_0) = (-1, \pi)$ is given by the vector $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$?

Exercise 2

(b) (additional space for your solution)

Exercise 2

(c) (3 points)

Determine a basis of the following vector space:

$$V = \{\, \mathbf{x} \in \mathbb{R}^5 \mid x_1 + x_2 = 0 \;\wedge\; x_1 + 5\,x_2 - 2\,x_3 = 0 \;\wedge\; x_1 - x_3 + x_5 = 0 \,\}.$$

Exercise 2

(d) (6 points)

Given is the matrix

$$M = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 2 & 0 \\ -5\,a & 5 & 4 \end{pmatrix},\ a \in \mathbb{R}.$$

Show that the matrix M has the eigenvalue $\lambda = 3$.

Compute the eigenspace of M associated to the eigenvalue $\lambda = 3$.

Compute the other eigenvalues of M.

Exercise 2

(d) (additional space for your solution)

Exercise 2

(e) (8 points)

Given is the system of linear equations

$$\begin{array}{rcrcrcl} 4\,x_1 & & & + & 4\,x_3 & = & 8 \\ 12\,x_1 & + & 16\,x_2 & + & 8\,x_3 & = & 36+4r \\ & & x_2 & - & 2\,x_3 & = & s \\ 3\,x_1 & + & 2\,x_2 & + & 3\,x_3 & = & 9-2r+s \\ x_1 & + & x_2 & + & x_3 & = & 3 \end{array} \quad .$$

For which values of the parameters $r \in \mathbb{R}$, $s \in \mathbb{R}$ is the system of equations solvable?

Determine the corresponding solution of the system.

Exercise 2

(e) (additional space for your solution)

Part II: Multiple-choice Question (50 points)

General instructions for multiple-choice questions:

(i) The solution must be reported in the multiple-choice solution form, which is distributed together with the exam. Only the answers reported in the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility can be marked.

(iii) When two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions.

Exercise 3 (25 points)

Question 1 (3 points)

The function $f(x, y) = -x$ has its minimum under the constraint $\varphi(x, y) = \frac{x^2}{4} + \frac{y^2}{25} - 1 = 0$ at

(a) $P = (-2, 0)$.

(b) $P = (0, 5)$.

(c) $P = (2, 0)$.

(d) $P = (3, 1)$.

Exercise 3

Question 2 (3 points)

Given is the function

$$f(x) = \begin{cases} a\,(x-2)^2 - 1 & \text{for } 2 \leq x \leq 4 \\ 0 & \text{elsewhere} \end{cases}.$$

(a) f is a density function for all $a \in \mathbb{R}$.

(b) f is a density function only for $a = \frac{9}{8}$.

(c) f is a density function only for $a = -\frac{8}{9}$.

(d) f is a density function for no $a \in \mathbb{R}$.

Exercise 3

Question 3 (3 points)

$A = (a_{ij})$ is a 6×4 matrix. It follows that:

(a) The rank of A^T is at least 4.

(b) The rank of A^T is less than 5.

(c) The rank of A^T is at most 6.

(d) The matrix A^T has at least one regular 4×4 sub-matrix.

Exercise 3

Question 4 (2 points)

A and B are square matrices with $\det(A) = 2$ and $\det(B) = -3$; the matrix C is defined by $C = A^2 + 3\,B$. It follows that:

(a) $\det(C) = 13$.

(b) $\det(C) = -5$.

(c) $\det(C) = -36$.

(d) It is not possible to make a general statement on $\det(C)$.

Exercise 3

Question 5 (4 points)

Given are the vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} -1 \\ 4 \\ t \end{pmatrix}.$$

(a) Only for $t = 6$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) Only for $t = 6$ and $t = 0$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(c) For all $t \in \mathbb{R}$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(d) For no $t \in \mathbb{R}$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Exercise 3

Question 6 (2 points)

A is a 5×4 matrix and $rg(A; \mathbf{b}) = 4$. It follows that:

(a) The system of linear equations $A\,\mathbf{x} = \mathbf{b}$ has no solution.

(b) The system of linear equations $A\,\mathbf{x} = \mathbf{b}$ has no solution or a unique solution.

(c) The system of linear equations $A\,\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(d) None of the above answers is correct.

Exercise 3

Question 7 (4 points)

The indefinite integral

$$\int \frac{x}{5\,x+2}\,dx,\ (x>0)$$

is

(a) $\frac{x}{5} - \frac{2}{25}\,\ln(5\,x+2) + C.$

(b) $\frac{\frac{1}{2}\,x^2}{\frac{5}{2}\,x^2+2\,x} + C.$

(c) $\frac{1}{2}\,x^2\,\ln(5\,x+2) + C.$

(d) None of the above answers is correct.

Exercise 3

Question 8 (4 points)

Given are the matrix A and the vector $\mathbf{b}$:

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$

It follows that:

(a) $A^n \mathbf{b} = \mathbf{b}$ for all $n \in \mathbb{N}$.

(b) $A^n \mathbf{b} = 2^n \mathbf{b}$ for all $n \in \mathbb{N}$.

(c) $A^n \mathbf{b} = 3^n \mathbf{b}$ for all $n \in \mathbb{N}$.

(d) None of the above answers is correct.

Exercise 4 (25 points)

Question 1 (3 points)

The improper integral

$$\int_{-\infty}^{1} e^{2\,x+1}\,dx$$

has the value

(a) e^2.

(b) $\frac{e^3}{2}$.

(c) $\frac{e^4}{5}$.

(d) The improper integral does not exist.

Exercise 4

Question 2 (3 points)

Let $f : \mathbb{R} \times \mathbb{R}_{++}$ be the function of two variables defined by

$$f(x, y) = x^2 - \ln(y).$$

For which value of $t \in \mathbb{R}_{++}$ is the gradient of f at the point (t, t) orthogonal to the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

(a) $t = \sqrt{0.5}$ and $t = -\sqrt{0.5}$.

(b) $t = \frac{1}{2}$.

(c) $t = \frac{\sqrt{2}}{2}$.

(d) There is no $t \in \mathbb{R}_{++}$ such that $\mathbf{grad} f(t, t)$ is orthogonal to $\mathbf{b}$.

Exercise 4

Question 3 (4 points)

The 4×5 matrix

$$A = \begin{pmatrix} 1 & 1 & -3 & 5 & 6 \\ -2 & -1 & 2 & 4 & 9 \\ 9 & 5 & -10 & 0 & 1 \\ 2 & 0 & 3 & -7 & 1 \end{pmatrix}$$

(a) has rank 2.

(b) has rank 3.

(c) has rank 4.

(d) has rank 5.

Exercise 4

Question 4 (4 points)

Consider the 2×2 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

It follows that:

(a) $A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

(b) $A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.

(c) $A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & -1 \\ 1 & -1 & 1 \end{pmatrix}$.

(d) None of the above answers is correct.

Exercise 4

Question 5 (2 points)

The matrix

$$A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

has the eigenvector

(a) $\mathbf{b} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$.

(b) $\mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

(c) $\mathbf{b} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$.

(d) $\mathbf{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Exercise 4

Question 6 (3 points)

Consider the initial value problem

$$\begin{aligned} &(1-a)\, y_{k+1} + a\, y_k - 1 = 0, \text{ where } a \neq -1,\ a \neq 0 \\ &y_0 = 2 \end{aligned} \quad .$$

For which value of the parameter a do we have $y_1 = \frac{5}{3}$?

(a) $a = 2$.

(b) $a = -2$.

(c) $a = \frac{1}{2}$.

(d) None of the above answers is correct.

Exercise 4

Question 7 (2 points)

The general solution of the difference equation

$$3\,(y_k - 2\,y_{k+1}) = 6 - 7\,y_k$$

is

(a) oscillating and convergent.

(b) oscillating and divergent.

(c) monotone and convergent.

(d) monotone and divergent.

Exercise 4

Question 8 (4 points)

The general solution of the difference equation

$$(2-c)\,y_{k+1} + (1+c)\,y_k = 2,$$

with $c \in \mathbb{R} \setminus \{2\}$, is oscillating and divergent if and only if

(a) $c > 2$.

(b) $c \in [\frac{1}{2}, 2)$.

(c) $c < -\frac{1}{2}$.

(d) The general solution of the difference equation is for no $c \in \mathbb{R} \setminus \{2\}$ oscillating and divergent.

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Multiple-choice answer sheet

Exercise 3 (25 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

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Exercise 4 (25 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐