Mathematics A
Master Solutions Exam Autumn Semester 2017

Prof. Dr. Enrico De Giorgi*

30 January 2018

Part I: Open questions (50 points)

General instructions for open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain only a single version.

(vi) Temporary computations or sketches must be reported in separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (26 points)

(a1) (4 points)

Given is the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = \ln\left(\sqrt{x-2} - 4\right) + \ln\left(\sqrt{x-2} + 4\right).$$

Determine the domain D_f and the range R_f of f.

Hint: Simplify the logarithm terms first.

Solution:

First of all, we apply the property $\ln(a) + \ln(b) = \ln(a\,b)$ (for $a, b > 0$) to simplify the terms of the logarithm and we obtain:

$$\begin{aligned} y &= \ln\left(\sqrt{x-2} - 4\right) + \ln\left(\sqrt{x-2} + 4\right) \\ &= \ln\left(\left(\sqrt{x-2} - 4\right)\ \left(\sqrt{x-2} + 4\right)\right) \\ &= \ln(x - 2 - 16) \\ &= \ln(x - 18). \end{aligned}$$

Therefore, the function f is defined if and only if $x - 18 > 0$, i.e., $x > 18$. We check that for $x > 18$ both summands in the first line are well defined, hence the simplification (line 1 to line 2) is legitimate. It follows that:

$$D_f = (18, \infty).$$

Moreover,

$$x \in D_f \Leftrightarrow x > 18 \Leftrightarrow f(x) = \ln(x - 18) \in \mathbb{R}.$$

It follows that:

$$R_f = \mathbb{R}.$$

Alternatively, if the property $\ln(a) + \ln(b) = \ln(a\,b)$ is not applied, the conditions for the domain of f are:

$$x \in D_f \Leftrightarrow \begin{cases} x - 2 \geq 0 \\ \sqrt{x-2} - 4 > 0 \\ \sqrt{x-2} + 4 > 0 \end{cases} \Leftrightarrow \sqrt{x-2} - 4 > 0 \Leftrightarrow x - 2 > 16 \Leftrightarrow x > 18.$$

Exercise 1

(a2) (3 points)

Given is the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = \ln\left(\sqrt{x-2} - 4\right) + \ln\left(\sqrt{x-2} + 4\right).$$

Is f strictly concave on its domain (proof)?

Solution:

We apply the following result: If f is twice differentiable and $f''(x) < 0$ for all $x \in (a, b)$, then f is strictly concave on (a, b).

Because in our case $f(x) = \ln(x - 18)$, then $f'(x) = \frac{1}{x-18}$ and $f''(x) = \frac{-1}{(x-18)^2}$. Therefore, $f''(x) < 0$ for all $x \in (18, \infty)$ and thus f is strictly concave on its domain.

Exercise 1

(a3) (3 points)

Given is the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = \ln\left(\sqrt{x-2}-4\right) + \ln\left(\sqrt{x-2}+4\right).$$

Determine the inverse function f^{-1} of f.

Solution:

For $y \in R_f = \mathbb{R}$ we have:

$$y = \ln(x-18) \Leftrightarrow e^y = x - 18 \Leftrightarrow e^y + 18 = x.$$

Because $D_{f^{-1}} = R_f = \mathbb{R}$ and $R_{f^{-1}} = D_f = (18, \infty)$ it follows that:

$$f^{-1} : \mathbb{R} \to (18, \infty), \quad x \mapsto y = f^{-1}(x) = e^x + 18.$$

Exercise 1

(b) (6 points)

A start-up borrows $1,000,000$ CHF to finance its activity. The bank agrees on a lower annual interest rate of 0.5% during the first 5 years, during which the start-up has to pay $10,000$ CHF at the end of each year. Afterwards, the interest rate increases to 2% and the start-up will pay C^I CHF at the end of each year. The plan is to repay the loan in 15 years.

What is the payment C^I such that the plan of the start-up is feasible?

Solution:

The constant payments of 10,000 CHF at the end of each year for the *first* 5 years correspond to a 5-year annuity immediate with end value after 5 years given by

$$A_5 = 10,000\,\frac{(1+0.5\%)^5 - 1}{0.5\%} \approx 50,502.50 \text{ (CHF)}.$$

The constant payments of C^I CHF at the end of each year for the *next* 10 years correspond to a 10-year annuity immediate with end value after 15 years given by

$$A_{10} = C^I \cdot \frac{(1+2.0\%)^{10} - 1}{2.0\%}.$$

The present value at time 0 of A_5 and A_{10} must correspond to 1,000,000 CHF, i.e.,

$$1,000,000 = \frac{A_5}{(1+0.5\%)^5} + \frac{A_{10}}{(1+0.5\%)^5\,(1+2.0\%)^{10}}$$

$$\Leftrightarrow \quad 1,000,000 \cdot (1+0.5\%)^5 = A_5 + \frac{A_{10}}{(1+2.0\%)^{10}}$$

$$\Leftrightarrow \quad 1,000,000 \cdot (1+0.5\%)^5 = 10,000 \cdot \frac{(1+0.5\%)^5 - 1}{0.5\%} + \frac{C^I}{(1+2.0\%)^{10}} \cdot \frac{(1+2.0\%)^{10} - 1}{2.0\%}$$

$$\Leftrightarrow \quad \frac{C^I}{(1+2.0\%)^{10}} \cdot \frac{(1+2.0\%)^{10} - 1}{2.0\%} = 1,000,000 \cdot (1+0.5\%)^5 - 10,000 \cdot \frac{(1+0.5\%)^5 - 1}{0.5\%}$$

$$\Leftrightarrow \quad C^I = \frac{1,000,000 \cdot (1+0.5\%)^5 - 10,000 \cdot \frac{(1+0.5\%)^5-1}{0.5\%}}{\frac{(1+2.0\%)^{10}-1}{2.0\%}}\,(1+2.0\%)^{10}$$

$$\Leftrightarrow \quad C^I \approx 108,515.40 \text{ (CHF)}.$$

Exercise 1

(c) (4 points)

Compute

$$\lim_{x\to 0} \frac{2}{x^4}\, e^{-\frac{1}{x^2}}.$$

Solution:

We have:

$$\begin{aligned}
\lim_{x\to 0} \frac{2}{x^4}\, e^{-\frac{1}{x^2}} &\overset{y=\frac{1}{x^2}}{=} \lim_{y\to\infty} 2\, y^2\, e^{-y} \\
&= \lim_{y\to\infty} \frac{2\, y^2}{e^y} \\
&\overset{\text{de l'Hôpital}}{=} \lim_{y\to\infty} \frac{4\, y}{e^y} \\
&\overset{\text{de l'Hôpital}}{=} \lim_{y\to\infty} \frac{4}{e^y} \\
&= 0.
\end{aligned}$$

Exercise 1

(d) (6 points)

A professional runner runs 20 kilometers during the first hour. Afterwards, her performance decreases by a factor $a \in (0,1]$ for each additional hour of running, e.g., in the second hour she runs $20 * (1-a)$ kilometers. For which values of $a \in (0,1]$ and $b \geq 20$ does the runner complete a competition of b kilometers if she can run as long as she wants?

Represent graphically the solution set in an (a,b)-system.

Solution:

Let a_n be the number of kilometers run during the n-th hour. We have:

$$\begin{aligned} a_1 &= 20, \\ a_n &= a_{n-1}(1-a), \quad n = 2, 3, \ldots \end{aligned}$$

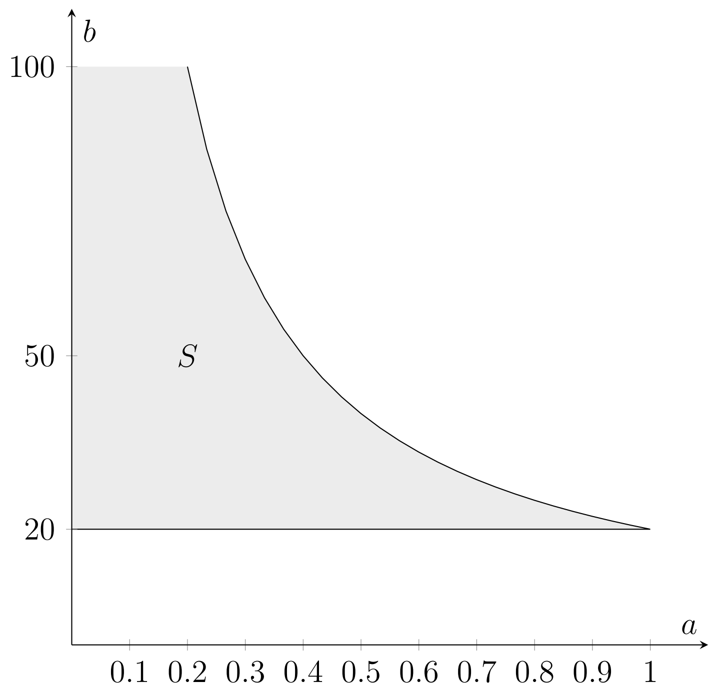
i.e., $\{a_n\}_{n \in \mathbb{N}}$ is a geometric sequence with $a_1 = 20$ and $q = (1-a)$.

The condition such that the runner completes a competition of b kilometers if she can run as long as she wants corresponds to:

$$\sum_{n=1}^{\infty} a_n \geq b \Leftrightarrow \frac{a_1}{1-q} \geq b \Leftrightarrow \frac{20}{1-(1-a)} \geq b \Leftrightarrow \frac{20}{a} \geq b.$$

Therefore, the solution set is:

$$S = \left\{ (a,b) \in (0,1] \times [20, \infty) \, : \, b \leq \frac{20}{a} \right\}.$$



Exercise 2 (24 points)

(a1) (5 points)

Let $a_k = \ln\left(1 + \left(\frac{1}{2}\right)^k\right)$ for $k = 1, 2, \ldots$

Use the second order Taylor polynomial P_2 of the function

$$f \,:\, D_f \to \mathbb{R},\, x \mapsto y = f(x) = \ln(1+x)$$

at $x_0 = 0$ to compute an approximation of $\sum_{k=1}^{\infty} a_k$.

Solution:

The second order Taylor polynomial of f at x_0 is given by:

$$P_2(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2}\,(x - x_0)^2.$$

With $x_0 = 0$ we have:

$$\begin{aligned}
f(x_0) &= f(0) = \ln(1+0) = 0 \\
f'(x) &= \frac{1}{1+x} \Rightarrow f'(x_0) = f'(0) = \frac{1}{1} = 1 \\
f''(x) &= -\frac{1}{(1+x)^2} \Rightarrow f''(x_0) = f''(0) = -\frac{1}{1^2} = -1.
\end{aligned}$$

It follows that:

$$P_2(x) = x - \frac{1}{2}\,x^2.$$

Therefore,

$$a_k = \ln\left(1 + \left(\frac{1}{2}\right)^k\right) \approx P_2\left(\left(\frac{1}{2}\right)^k\right) = \left(\frac{1}{2}\right)^k - \frac{1}{2}\left(\frac{1}{2}\right)^{2k} = \left(\frac{1}{2}\right)^k - \frac{1}{2}\left(\frac{1}{4}\right)^k.$$

It follows that:

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k &\approx \sum_{k=1}^{\infty}\left[\left(\frac{1}{2}\right)^k - \frac{1}{2}\left(\frac{1}{4}\right)^k\right] \\
&= \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k - \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k \\
&= \frac{1}{2}\,\frac{1}{1-\frac{1}{2}} - \frac{1}{2}\cdot\frac{1}{4}\,\frac{1}{1-\frac{1}{4}} \\
&= \frac{1}{2}\cdot 2 - \frac{1}{8}\cdot\frac{4}{3} \\
&= 1 - \frac{1}{6} \\
&= \frac{5}{6}.
\end{aligned}$$

Exercise 2

(a2) (4 points)

Given is the function

$$f\,:\,D_f \to \mathbb{R}, \quad x \mapsto y = f(x) = \ln(1+x).$$

R_2 denotes the second order remainder term of f at $x_0 = 0$.

Show that

$$\sum_{k=1}^{\infty} R_2\left(\left(\frac{1}{2}\right)^k\right) \leq \frac{1}{21}.$$

Solution:

According to Taylor's Theorem we have:

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!}\,x^3$$

where $\xi \in [0, x]$.

We have:

$$f^{(3)}(x) = \frac{2}{(1+x)^3}.$$

Therefore, for $x \in [0,1]$,

$$|R_2(x)| = \frac{\left|f^{(3)}(\xi)\right|}{3!}\,|x|^3 = \frac{2}{3!}\,\underbrace{\frac{1}{(1+\xi)^3}}_{\leq 1}\,x^3 \leq \frac{1}{3}\,x^3.$$

If follows that:

$$\sum_{k=1}^{\infty} R_2\left(\left(\frac{1}{2}\right)^k\right) \leq \sum_{k=1}^{\infty} \frac{1}{3}\left[\left(\frac{1}{2}\right)^k\right]^3 = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{8}\right)^k = \frac{1}{3}\cdot\frac{1}{8}\cdot\frac{1}{1-\frac{1}{8}} = \frac{1}{3}\cdot\frac{1}{7} = \frac{1}{21}.$$

Exercise 2

(b) (4 points)

Given is the function

$$f(x,y) = \frac{\ln(9 - 9\,x^2 - y^2)}{(x-y)\,\sqrt{4 - x^2 - y^2}}.$$

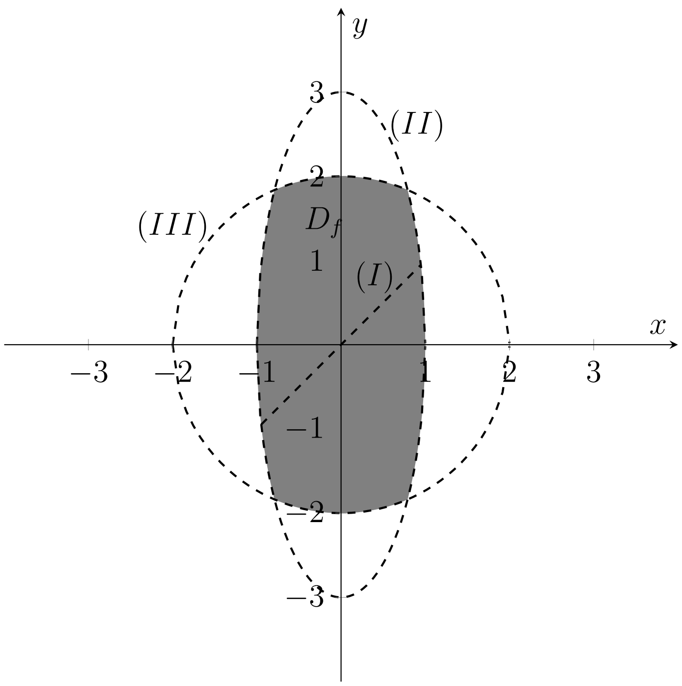
Determine the domain D_f of f and represent it graphically.

Solution:

We have:

$$x \in D_f \Leftrightarrow \begin{cases} x - y \neq 0 \\ 9 - 9\,x^2 - y^2 > 0 \\ 4 - x^2 - y^2 > 0 \end{cases} \Leftrightarrow \begin{cases} x \neq y \ (I) \\ x^2 + \frac{y^2}{3^2} < 1 \ (II) \\ x^2 + y^2 < 2^2 \ (III) \end{cases} .$$

This corresponds to the area inside the ellipse (II) with centre $(0,0)$ and semi-axis $a = 1$ and $b = 3$ intersected with the area inside the circle (III) with centre $(0,0)$ and radius 2, and then excluded all points on the line $y = x$ (I). The line $y = x$ intersects the ellipse when $x^2 + \frac{x^2}{9} = 1$, i.e., $x^2 = \frac{9}{10}$ or $x = \pm\frac{3}{\sqrt{10}}$. The following figure illustrates the domain of f:



Exercise 2

(c) (5 points)

Given is the utility function
$$u(c_1, c_2) = c_1^{\alpha}\, c_2^{1-\alpha}$$
for $\alpha \in (0, 1)$, where c_1, c_2 are the units of consumption of goods 1 and 2, and the budget constraint
$$C \;:\; p_1\, c_1 + p_2\, c_2 = 10$$
for prices $p_1 > 0$ and $p_2 > 0$.

For which values of the parameters α, p_1, p_2 does the contour line $u(c_1, c_2) = \sqrt{2}$ touches the budget line C at the consumption bundle $(c_1^\star, c_2^\star) = (1, 2)$?

Solution:

The parameters α, p_1, p_2 are such that the following three conditions hold:

(i) The consumption bundle $(c_1^\star, c_2^\star) = (1, 2)$ belongs to the budget line, i.e., $p_1 + 2\, p_2 = 10$. It follows that $p_1 = 10 - 2\, p_2$.

(ii) The consumption bundle $(c_1^\star, c_2^\star) = (1, 2)$ has utility $\sqrt{2}$, i.e., $u(c_1^\star, c_2^\star) = 1^{\alpha}\, 2^{1-\alpha} = \sqrt{2}$. It follows that $\alpha = 0.5$.

(iii) The curves $f(c_1, c_2) = u(c_1, c_2) - \sqrt{2} = c_1^{0.5}\, c_2^{0.5} - \sqrt{2} = 0$ and $\varphi(c_1, c_2) = p_1\, c_1 + p_2\, c_2 - 10 = 0$ touch at $(c_1^\star, c_2^\star) = (1, 2)$, i.e. (implicit function theorem),

$$\begin{aligned}
-\frac{f_{c_1}(1,2)}{f_{c_2}(1,2)} &= -\frac{\varphi_{c_1}(1,2)}{\varphi_{c_2}(1,2)} \\
\overset{\alpha=0.5}{\Leftrightarrow} \quad -\frac{0.5\, c_1^{-0.5}\, c_2^{0.5}}{0.5\, c_1^{0.5}\, c_2^{-0.5}}\Bigg|_{(c_1,c_1)=(1,2)} &= -\frac{p_1}{p_2} \\
\overset{p_1=10-2\,p_2}{\Leftrightarrow} \quad 2 &= \frac{10 - 2\, p_2}{p_2} \\
\Leftrightarrow \quad 2\, p_2 &= 10 - 2\, p_2 \\
\Leftrightarrow \quad p_2 &= \frac{5}{2}.
\end{aligned}$$

Therefore, $p_1 = 10 - 2 \cdot \frac{5}{2} = 5$, $p_2 = \frac{5}{2}$ and $\alpha = 0.5$.

Exercise 2

(d) (6 points)

The functions f and g are defined on $\mathbb{R}^2_{++}$ and have range $\mathbb{R}_{++}$. Moreover, the function f is homogeneous of degree r and the function g is homogeneous of degree $r-2$. The function h satisfies:

$$h(x,y) = \frac{f(x,y)}{g(x,y)},$$

$$h_y(x,y) = x - \frac{3}{2}\,x^{0.5}\,y^{0.5}$$

and

$$\varepsilon_{h,x}(x,y) = \frac{x\,y - \frac{1}{2}\,x^{0.5}\,y^{1.5}}{x\,y - x^{0.5}\,y^{1.5}}$$

Determine h and simplify the terms of the function.

Solution:

Because f is homogeneous of degree r and g is homogeneous of degree $r-2$, then

$$h(\lambda\,x,\lambda\,y) = \frac{f(\lambda\,x,\lambda\,y)}{g(\lambda\,x,\lambda\,y)} = \frac{\lambda^r\,f(x,y)}{\lambda^{r-2}\,g(x,y)} = \lambda^2\,\frac{f(x,y)}{g(x,y)} = \lambda^2\,h(x,y),$$

i.e., h is homogeneous of degree 2. Therefore, the Euler relation implies:

$$x\,h_x(x,y) + y\,h_y(x,y) = 2\,h(x,y).$$

Because

$$\varepsilon_{h,x}(x,y) = \frac{x}{h(x,y)}\,h_x(x,y)$$

we obtain:

$$\varepsilon_{h,x}(x,y)\,h(x,y) + y\,h_y(x,y) = 2\,h(x,y),$$

i.e.,

$$h(x,y) = \frac{y\,h_y(x,y)}{2 - \varepsilon_{h,x}(x,y)}.$$

We plug the expressions for $h_y(x,y)$ and $\varepsilon_{h,x}$ in this latter equation and obtain:

$$\begin{aligned}
h(x,y) &= \frac{y\,\left(x - \frac{3}{2}\,x^{0.5}\,y^{0.5}\right)}{2 - \frac{x\,y - \frac{1}{2}\,x^{0.5}\,y^{1.5}}{x\,y - x^{0.5}\,y^{1.5}}} \\
&= \frac{x\,y - \frac{3}{2}\,x^{0.5}\,y^{1.5}}{2\,x\,y - 2\,x^{0.5}\,y^{1.5} - x\,y + \frac{1}{2}\,x^{0.5}\,y^{1.5}}\,\left(x\,y - x^{0.5}\,y^{1.5}\right) \\
&= \frac{x\,y - \frac{3}{2}\,x^{0.5}\,y^{1.5}}{x\,y - \frac{3}{2}\,x^{0.5}\,y^{1.5}}\,\left(x\,y - x^{0.5}\,y^{1.5}\right) \\
&= x\,y - x^{0.5}\,y^{1.5}.
\end{aligned}$$

It follows that:

$$h(x,y) = x\,y - x^{0.5}\,y^{1.5}.$$

(d) (Solution continued)

The following two solution attempts are incomplete and have been awarded with 1 point:

(i) Since we have $\varepsilon_{h,x}(x,y) = \frac{x}{h(x,y)}\, h_x(x,y)$, it has to follow that $h(x,y) = x\,y - x^{0.5}\,y^{1.5}$.
Explanation: The fraction $\varepsilon_{h,x}(x,y) = \frac{x\,y - \frac{1}{2}\,x^{0.5}\,y^{1.5}}{x\,y - x^{0.5}\,y^{1.5}}$ may be a result of cancellation. In fact, the task was to show that actually $h(x,y) = x\,y - x^{0.5}\,y^{1.5}$ holds.

(ii) Integrate $h_y(x,y) = x - \frac{3}{2}\,x^{0.5}\,y^{0.5}$ with respect to y gives $h(x,y) = x\,y - x^{0.5}\,y^{1.5}$.
Explanation: Integrating $h_y(x,y)$ with respect to y delivers the indefinite integral $H(x,y) = x\,y - x^{0.5}\,y^{1.5} + C(x)$, where $C(x)$ is an indefinite expression depending on x. The actual task was to show that $C(x) = 0$ holds.

Part II: Multiple-choice question (50 points)

General instructions for multiple-choice questions:

(i) The solution must be reported in the multiple-choice solution form. Only the answers reported in the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility can be marked.

(iii) When two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions.

Exercise 3 (24 points)

Question 1 (4 points)

Let A and B be two propositions. The compound proposition $A \vee (\neg A \Rightarrow B)$ is equivalent to

(a) A.

(b) B.

(c) $A \vee B$.

(d) $A \wedge B$.

Solution:

Answer is (c). The following truth tables applies:

A	T	T	F	F
B	T	F	T	F
$\neg A$	F	F	T	T
$\neg A \Rightarrow B$	T	T	T	F
$A \vee (\neg A \Rightarrow B)$	**T**	**T**	**T**	**F**
$A \vee B$	**T**	**T**	**T**	**F**
$A \wedge B$	T	F	F	F

Therefore, $A \vee (\neg A \Rightarrow B)$ is equivalent to $A \vee B$.

Exercise 3

Question 2 (3 points)

Let f be a continuous function. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence with $a_n \in D_f$ for all $n \in \mathbb{N}$ that is monotone and convergent. The sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n)$ for all $n \in \mathbb{N}$ is

(a) convergent.

(b) divergent.

(c) monotone.

(d) None of the above answers is correct.

Solution:

Answer is (d). We have:

- (a) is wrong. Take as an example $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ and $f(x) = \frac{1}{x}$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and convergent and f is continuous on $(0, \infty)$. However, the sequence $\{b_b\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n) = \frac{1}{\frac{1}{n}} = n$ is divergent.
- (b) is wrong. Take as an example $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ and $f(x) = x^2$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and convergent and f is continuous on $\mathbb{R}$. However, the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n) = \left(\frac{1}{n}\right)^2 = \frac{1}{n^2}$ is monotone and convergent.
- (c) is wrong. Take as an example $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ and $f(x) = \sin(2\,\pi\,x)$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotone and convergent and f is continuous on $\mathbb{R}$. However, the sequence $\{b_n\}_{n\in\mathbb{N}}$ defined by $b_n = f(a_n) = \sin\left(\frac{2\,\pi}{n}\right)$ is not monotone.

Exercise 3

Question 3 (2 points)

Which of the following statements about a function f and a point $x_0 \in D_f$ is correct?

(a) If f is continuous in x_0 then it is also differentiable in x_0.

(b) If f is differentiable in x_0 then it is also continuous in x_0.

(c) f is differentiable in x_0 if and only if it is continuous in x_0.

(d) If f is differentiable in x_0 then it is also discontinuous in x_0.

Solution:

Answer is (b). If f is differentiable at $x_0 \in D_f$ then it is also continuous at $x_0 \in D_f$. Consequently (b) is correct and (d) is wrong. That (a) and (c) are wrong can be seen with the help of the following example: $f(x) = |x|$ with domain $D_f = \mathbb{R}$ and $x_0 = 0$. The absolute value function is overall continuous in $\mathbb{R}$, but not differentiable at $x_0 = 0$.

Exercise 3

Question 4 (3 points)

An investor can choose between two projects. Project I requires an initial investment of $100,000$ CHF and returns $50,000$ CHF in 6 months and $60,000$ CHF in 1 year. Project II requires an initial investment of $100,000$ CHF and returns $110,000$ CHF in 1 year.

(a) Project I should be preferred to Project II if the interest rate is strictly positive.

(b) Project II should be preferred to Project I if the interest rate is strictly positive.

(c) Project I and Project II have the same net present value.

(d) Whether Project I should be preferred to Project II or Project II should be preferred to Project I depends on the value of the strictly positive interest rate.

Solution:

Answer is (a). The present value of project I is $-100,000+\frac{50,000}{(1+i)}+\frac{60,000}{(1+i)^2}$. The present value of project II is $-100,000+\frac{110,000}{(1+i)^2}$. Therefore, project I is preferred to project II when

$$-100,000+\frac{50,000}{(1+i)}+\frac{60,000}{(1+i)^2}>-100,000+\frac{110,000}{(1+i)^2}\Leftrightarrow\frac{50,000}{(1+i)}>\frac{50,000}{(1+i)^2}\Leftrightarrow 1>\frac{1}{1+i}.$$

The latter is always satisfied for $i>0$.
Without explicitly computing the present values, one can also argue that Project 1 is better because it requires the same investment of Project II, returns 60,000 CHF in year 2, but 50,000 CHF already in year 1. Therefore, the total amount returned by Project I in year 2 is strictly larger than 110,000 CHF if the interest rate is strictly positive.

Exercise 3

Question 5 (3 points)

A financial advisor suggests to his client two options to repay a mortgage: option 1 is to repay the mortgage with constant payments C^D at the beginning of each year for n^D years, while option 2 is to repay the same mortgage with constant payments C^I at the end of each year for n^I years.

Assuming that the interest rate is strictly positive, it follows that:

(a) $C^I = C^D$ when $n^I = n^D$.

(b) $C^I > C^D$ when $n^I = n^D$.

(c) $C^I < C^D$ when $n^I = n^D$.

(d) $C^I < C^D$ if and only if $n^I > n^D$.

Solution:

Answer is (b). If $n^I = n^D$ and the interest rate is strictly positive, then interests generated by payments at the beginning of the year are strictly larger than interests generated by payments at the end of the year. This holds simply because with payments at the beginning of the year interests are generated for one additional year compared to payments at the end of the year. The client has to pay less if payments are at the beginning of the year, because she can profit from higher interests generated by her payments. If payments at the end of the year last longer (i.e., $n^I > n^D$), then it is generally not true that yearly payments C^I are smaller than yearly payments C^D. For example, to repay a loan of 1,000,000 CHF over 20 years, the constant payments at the beginning of each year must correspond to $C^D = 76,421.50$ CHF if the interest rate is $i = 5\%$. To repay the same loan over 21 years, the constant payments at the end of each year must correspond to $C^I = 77,996.10$ CHF. Therefore, $C^I > C^D$ despite $n^I > n^D$.

Exercise 3

Question 6 (3 points)

We consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = \begin{cases} \frac{\sin(x)}{a\,(x-\pi)} & \text{for } x \neq \pi \\ a & \text{for } x = \pi \end{cases}.$$

For which value of $a \in \mathbb{R}$ is f overall continuous?

(a) $a = 1$.

(b) $a = -1$.

(c) $a \in \{-1, 1\}$.

(d) f is for no $a \in \mathbb{R}$ overall continuous.

Solution:

Answer is (d). Clearly, f is continuous for $x \neq \pi$. Moreover,

$$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} \frac{\sin(x)}{a\,(x-\pi)} \overset{\text{de l'Hôpital}}{=} \lim_{x\to\pi} \frac{\cos(x)}{a} = -\frac{1}{a}.$$

It follows that f is continuous in $x = \pi$ if and only if $-\frac{1}{a} = a$, i.e., $a^2 = -1$. This latter equation does not have solutions on $\mathbb{R}$.

Exercise 3

Question 7 (3 points)

Let $f(x) = 1 + 3\,x - 4\,x^4$ and P_4 the forth order Taylor polynomial of f at $x_0 = 1$. Which of the following statements on the fourth order reminder term R_4 at $x_0 = 1$ is correct?

(a) $R_4(x) > 0$ for all x.

(b) $R_4(x) < 0$ for all x.

(c) $R_4(x) = 0$ for all x.

(d) All cases $R_4(x) > 0$, $R_4(x) < 0$ and $R_4(x) = 0$ are possible for some $x \in \mathbb{R}$.

Solution:

Answer is (c). The reminder term $R_4(x)$ at $x_0 = 1$ is equal to zero for all x because f is a already a polynomial function of order 4. Therefore, f and P_4 coincides. One can also apply Taylor Theorem: there exists ξ such that

$$R_4(x) = \frac{f^{(5)}(\xi)}{5!}\,(x-1)^5.$$

Because $f^{(5)}(\xi) = 0$ for all possible values of ξ, then $R_4(x) = 0$.

Exercise 3

Question 8 (3 points)

For a function f, the elasticity $\varepsilon_f(x)$ is:

$$\varepsilon_f(x) = x\,\ln(x) + e^{3\,x}.$$

Let g be a function defined by $g(x) = f(a\,x)$ for $a > 0$. It follows that:

(a) $\varepsilon_g(x) = x\,\ln(x) + e^{3\,x}$.

(b) $\varepsilon_g(x) = a\,x\,\ln(x) + a\,e^{3\,x}$.

(c) $\varepsilon_g(x) = \frac{x}{a}\,\ln(x) + \frac{e^{3\,x}}{a}$.

(d) None of the above answers is correct.

Solution:

Answer is (d). We have:

$$\begin{aligned}
\varepsilon_g(x) &= x\,\frac{g'(x)}{g(x)} \\
&= x\,\frac{f'(a\,x)\,a}{f(a\,x)} \\
&= (a\,x)\,\frac{f'(a\,x)}{f(a\,x)} \\
&= \varepsilon_f(a\,x) \\
&= a\,x\,\ln(a\,x) + e^{3\,a\,x}.
\end{aligned}$$

Exercise 4 (26 points)

Question 1 (3 points)

Given is the function

$$f : \mathbb{R}^2_{++} \to \mathbb{R}_{++}, \quad (x, y) \mapsto f(x, y) = (x^2 + 2\,x\,y + y^2)\,e^{x+y}.$$

Its partial elasticities $\varepsilon_{f,x}(x, y)$ and $\varepsilon_{f,y}(x, y)$ satisfy

(a) $\varepsilon_{f,x} > \varepsilon_{f,y}$ for all $(x, y) \in \mathbb{R}^2_{++}$.

(b) $\varepsilon_{f,x} < \varepsilon_{f,y}$ for all $(x, y) \in \mathbb{R}^2_{++}$.

(c) $\varepsilon_{f,x} < \varepsilon_{f,y}$ for $(x, y) \in \mathbb{R}^2_{++}$ with $x > y$.

(d) $\varepsilon_{f,x} < \varepsilon_{f,y}$ for $(x, y) \in \mathbb{R}^2_{++}$ with $x < y$.

Solution:

Answer is (d). First of all

$$f(x, y) = (x + y)^2 e^{x+y}$$

and

$$f_x(x, y) = \left[2\,(x + y) + (x + y)^2\right]\,e^{x+y} = f_y(x, y).$$

The following holds:

$$\begin{aligned}
&\varepsilon_{f,x}(x, y) < \varepsilon_{f,y}(x, y) \\
\Leftrightarrow \quad & x\,\frac{f_x(x, y)}{f(x, y)} < y\,\frac{f_y(x, y)}{f(x, y)} \\
\overset{f(x,y)>0}{\Leftrightarrow} \quad & x\,f_x(x, y) < y\,f_y(x, y) \\
\Leftrightarrow \quad & x\,\left[2\,(x + y) + (x + y)^2\right]\,e^{x+y} < y\,\left[2\,(x + y) + (x + y)^2\right]\,e^{x+y} \\
\overset{x,y>0}{\Leftrightarrow} \quad & x < y.
\end{aligned}$$

Therefore, (d) is correct.

Exercise 4

Question 2 (4 points)

Given is the function

$$f : \mathbb{R} \to \mathbb{R}_{++}, \quad x \mapsto f(x) = x^2 \, e^{x^2} + 1.$$

(a) f has a local maximum at $x_0 = 0$.

(b) f has a local minimum at $x_0 = 0$.

(c) f has an inflection point at $x_0 = 0$.

(d) f has no stationary points.

Solution:

Answer is (b). Because f is differentiable, a stationary point satisfies $f'(x) = 0$. We have:

$$f'(x) = 2\,x\,e^{x^2} + x^2\,e^{x^2}\,(2\,x) = 2\,x\,e^{x^2}\,(1 + x^2).$$

Therefore, $f'(x) = 0$ if and only if $x = 0$, i.e., $x = 0$ is a stationary point of f and (d) is wrong. Moreover,

$$f''(x) = (2 + 6\,x^2)\,e^{x^2} + (2\,x + 2\,x^3)\,e^{x^2}\,(2\,x) = 2\,e^{x^2}\,(1 + 5\,x^2 + 2\,x^4).$$

It follows that:

$$f''(0) = 2 > 0,$$

and $x = 0$ is a minimum.

Exercise 4

Question 3 (4 points)

Consider the function
$$f : D_f \to \mathbb{R},\ x \mapsto f(x) = \frac{1}{1+x}$$
and let P_3 and P_4 be the third and fourth order Taylor polynomials of f at $x_0 = 0$. It follows that:

(a) $P_3(x) > P_4(x)$ for all $x \in D_f \setminus \{x_0\}$.

(b) $P_3(x) < P_4(x)$ for all $x \in D_f \setminus \{x_0\}$.

(c) $P_3(x) = P_4(x)$ for all $x \in D_f \setminus \{x_0\}$.

(d) $P_3(x) > P_4(x)$, $P_3(x) < P_4(x)$ or $P_3(x) = P_4(x)$ are all possible for some $x \in D_f \setminus \{x_0\}$.

Solution:

Answer is (b). Because
$$P_4(x) = P_3(x) + \frac{f^{(4)}(0)}{4!}\, x^4,$$
then
$$P_3(x) < P_4(x) \Leftrightarrow \frac{f^{(4)}(0)}{4!}\, x^4 > 0 \Leftrightarrow f^{(4)}(0) > 0.$$
We have:
$$f'(x) = \frac{-1}{(1+x)^2}, \qquad f''(x) = \frac{2}{(1+x)^3}$$
$$f'''(x) = \frac{-6}{(1+x)^4}, \qquad f^{(4)}(x) = \frac{24}{(1+x)^5}$$

Therefore, $f^{(4)}(0) = 24 > 0$. It follows that $P_3(x) < P_4(x)$.

Exercise 4

Question 4 (4 points)

Given are the functions

$$f(x,y) = \sqrt{1 - 4\,x^2 - y^2}$$

and

$$g(x,y) = \ln\left(2\,x - x^2 - y^2 + 8\right),$$

and D_f and D_g are the corresponding domains.

We have:

(a) $D_f \subseteq D_g$.

(b) $D_g \subseteq D_f$.

(c) $D_f = D_g$.

(d) $D_f \cap D_g = \emptyset$.

Solution:

Answer is (a). For function f,

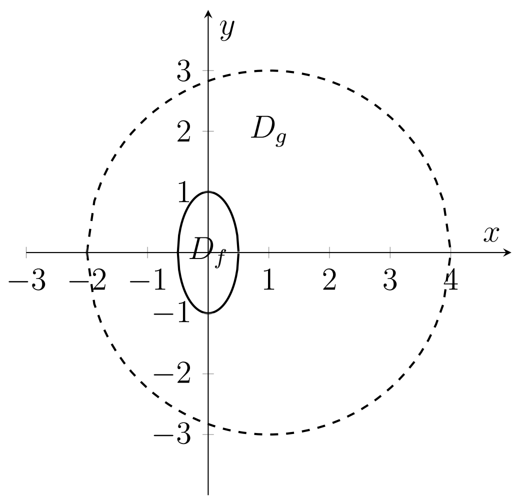
$$(x,y) \in D_f \Leftrightarrow 1 - 4\,x^2 - y^2 \geq 0 \Leftrightarrow \frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{1} \leq 1.$$

This corresponds to the area inside the ellipse with centre $(0,0)$ and semi-axes $a = \frac{1}{2}$ and $b = 1$.

For function g,

$$(x,y) \in D_g \Leftrightarrow 2\,x - x^2 - y^2 + 8 > 0 \Leftrightarrow x^2 - 2\,x + 1 + y^2 < 9 \Leftrightarrow (x-1)^2 + y^2 < 3^2.$$

This corresponds to the area inside the circle with centre $(1,0)$ and radius 3.



Exercise 4

Question 5 (3 points)

Let $f(x) = \sin(x)$ and P_3 its third order Taylor polynomial at $x_0 = 0$.

Which of the following statements concerning the third order reminder term R_3 of f at $x_0 = 0$ is correct?

(a) $|R_3(x)| \leq \frac{|x|^4}{128}$ for all $x \in \mathbb{R}$.

(b) $|R_3(x)| \leq \frac{|x|^4}{64}$ for all $x \in \mathbb{R}$.

(c) $|R_3(x)| \leq \frac{|x|^4}{32}$ for all $x \in \mathbb{R}$.

(d) $|R_3(x)| \leq \frac{|x|^4}{16}$ for all $x \in \mathbb{R}$.

Solution:

Answer is (d). Taylor theorem implies that:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}\, x^4.$$

Because $f^{(4)}(x) = \sin(x)$, then $|f^{(4)}(\xi)| \leq 1$ for all ξ and

$$|R_3(x)| = \frac{|f^{(4)}(\xi)|}{4!}\, |x|^4 \leq \frac{|x|^4}{24} \leq \frac{|x|^4}{16}.$$

Remark: logical thinking implies that only (d) can hold. Indeed, because only one answer is correct, then only (d) can be that answer. If (a) is true, then also (b), (c) and (d) are true. Similarly, if (b) is true then also (c) and (d) are true. Finally, if (c) is true then also (d) is true. Therefore, only (d) can be true.

Exercise 4

Question 6 (2 points)

Given is the function

$$f(x,y) = 8\left(\frac{1}{x} + \frac{1}{5\,y}\right)^{-0.5} + \sqrt{3\,x} + \sqrt{y} \quad (x > 0,\ y > 0).$$

(a) f is linear homogeneous.

(b) f is homogeneous of degree -0.5.

(c) f is homogeneous of degree 0.5.

(d) f is not homogeneous.

Solution:

Answer is (c). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\,x, \lambda\,y) &= 8\left(\frac{1}{\lambda\,x} + \frac{1}{5\,\lambda\,y}\right)^{-0.5} + \sqrt{3\,\lambda\,x} + \sqrt{\lambda\,y} \\
&= 8\left[\frac{1}{\lambda}\left(\frac{1}{x} + \frac{1}{5\,y}\right)\right]^{-0.5} + \lambda^{0.5}\sqrt{3\,x} + \lambda^{0.5}\sqrt{y} \\
&= 8\,\lambda^{0.5}\left(\frac{1}{x} + \frac{1}{5\,y}\right)^{-0.5} + \lambda^{0.5}\sqrt{3\,x} + \lambda^{0.5}\sqrt{y} \\
&= \lambda^{0.5}\left[8\left(\frac{1}{x} + \frac{1}{5\,y}\right)^{-0.5} + \sqrt{3\,x} + \sqrt{y}\right] \\
&= \lambda^{0.5}\,f(x,y).
\end{aligned}$$

It follows that f is homogenous of degree 0.5.

Exercise 4

Question 7 (3 points)

Given is the function

$$f(x,y) = \frac{x^2}{y} + 1 + \sqrt{x^2 + 5\,y^2} \quad (x > 0,\ y > 0)$$

and

$$g(x,y) = f(a\,x, a\,y),$$

where $a > 0$.

(a) g is linear homogeneous.

(b) g is homogeneous of degree a.

(c) g is homogeneous of degree $2\,a$.

(d) g is not homogeneous.

Solution:

Answer is (d). For $\lambda > 0$ we have:

$$\begin{aligned}
f(\lambda\,x, \lambda\,y) &= \left(\frac{(\lambda\,x)^2}{\lambda\,y} + 1\right) + \sqrt{(\lambda\,x)^2 + 5\,(\lambda\,y)^2} \\
&= \left(\frac{\lambda^2\,x^2}{\lambda\,y} + 1\right) + \sqrt{\lambda^2\,x^2 + 5\,\lambda^2\,y^2} \\
&= \left(\lambda\,\frac{x^2}{y} + 1\right) + \lambda\,\sqrt{x^2 + 5\,y^2} \\
&= \lambda\,\left(\frac{x^2}{y} + 1 + \sqrt{x^2 + 5\,y^2}\right) - \lambda + 1 \\
&= \lambda\,f(x,y) - \lambda + 1
\end{aligned}$$

It follows that:

$$g(\lambda\,x, \lambda\,y) = f(a\,\lambda\,x, a\,\lambda\,y) = f(\lambda\,a\,x, \lambda\,a\,y) = \lambda\,f(a\,x, a\,y) - \lambda + 1 = \lambda\,g(x,y) - \lambda + 1.$$

Therefore, g is not homogeneous.

Exercise 4

Question 8 (3 points)

Consider the function

$$f(x,y) = x^{a+1}\sqrt{y^{4\,a+4}} + (x\,y)^{\frac{3\,a+3}{2}} \qquad (x>0,\, y>0)$$

with $a \in \mathbb{R}$, and let $\varepsilon_{f,x}$ and $\varepsilon_{f,y}$ be its corresponding partial elasticities.

For which value of a does

$$\varepsilon_{f,x} + \varepsilon_{f,y} = 3$$

hold?

(a) $a = 0$.

(b) $a = 1$.

(c) $a = 2$.

(d) $a = 3$.

Solution:

Answer is (a). For $\lambda > 0$ we have:

$$\begin{aligned} f(\lambda\,x, \lambda\,y) &= (\lambda\,x)^{a+1}\sqrt{(\lambda\,y)^{4\,a+4}} + (\lambda\,x\,\lambda\,y)^{\frac{3\,a+3}{2}} \\ &= \lambda^{a+1}\,x^{a+1}\,\lambda^{\frac{4\,a+4}{2}}\sqrt{y^{4\,a+4}} + \lambda^{3\,a+3}\,(x\,y)^{\frac{3\,a+3}{2}} \\ &= \lambda^{a+1}\,x^{a+1}\,\lambda^{2\,a+2}\sqrt{y^{4\,a+4}} + \lambda^{3\,a+3}\,(x\,y)^{\frac{3\,a+3}{2}} \\ &= \lambda^{3\,a+3}\left[x^{a+1}\sqrt{y^{4\,a+4}} + (x\,y)^{\frac{3\,a+3}{2}}\right]. \end{aligned}$$

Therefore, f is homogeneous of degree $3\,a+3$ and Euler relation implies that:

$$\varepsilon_{f,x} + \varepsilon_{f,y} = 3\,a + 3.$$

It follows that:

$$\varepsilon_{f,x} + \varepsilon_{f,y} = 3 \Leftrightarrow a = 0.$$

Exams Assessment Level - Autumn Term 2017

1,202 Mathematics A

Multiple-choice answer sheet

Exercise 3 (24 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☒	☐
2.	☐	☐	☐	☒
3.	☐	☒	☐	☐
4.	☒	☐	☐	☐
5.	☐	☒	☐	☐
6.	☐	☐	☐	☒
7.	☐	☐	☒	☐
8.	☐	☐	☐	☒

Exercise 4 (26 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☒
2.	☐	☒	☐	☐
3.	☐	☒	☐	☐
4.	☒	☐	☐	☐
5.	☐	☐	☐	☒
6.	☐	☐	☒	☐
7.	☐	☐	☐	☒
8.	☒	☐	☐	☐