Mathematics B
Master Solution Spring Semester 2017

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Part I: Open Questions

Exercise 1

(a) (7 points)

The necessary conditions for maxima, minima or saddle points (x_0, y_0) of f are

$$\begin{cases} f_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) = 0 \end{cases}.$$

We thus compute the first order partial derivatives of f and we obtain

$$\begin{aligned} f_x(x, y) &= e^x + (x + y + a)\, e^x = (x + y + a + 1)\, e^x, \\ f_y(x, y) &= e^x - e^y. \end{aligned}$$

It follows that:

$$\begin{cases} f_x(x_0, y_0) = 0 \\ f_y(x_0, y_0) = 0 \end{cases} \Leftrightarrow \begin{cases} (x_0 + y_0 + a + 1)\, e^{x_0} = 0 \\ e^{x_0} - e^{y_0} = 0 \end{cases}.$$

From $e^{x_0} - e^{y_0} = 0$ we obtain

$$y_0 = x_0.$$

We plug this latter result into $(x_0 + y_0 + a + 1)\, e^{x_0} = 0$ and we find (using that $e^{x_0} \neq 0$ for all $x_0 \in \mathbb{R}$)

$$\begin{aligned} x_0 + y_0 + a + 1 = 0 \quad &\overset{x_0 = y_0}{\Leftrightarrow} \quad 2\, x_0 + a + 1 = 0 \\ &\Leftrightarrow \quad x_0 = -\frac{a + 1}{2}. \end{aligned}$$

Using that $y_0 = x_0$, we obtain a unique stationary point for f:

$$P = \left(-\frac{a + 1}{2}, -\frac{a + 1}{2} \right).$$

Next, we verify the sufficient conditions: if (x_0, y_0) satisfies the necessary conditions, then the following holds:

$$\begin{cases} f_{xx}(x_0, y_0) > 0 \\ f_{yy}(x_0, y_0) > 0 \\ f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0 \end{cases} \Rightarrow (x_0, y_0) \text{ is a minimum,}$$

$$\begin{cases} f_{xx}(x_0, y_0) < 0 \\ f_{yy}(x_0, y_0) < 0 \\ f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 > 0 \end{cases} \Rightarrow (x_0, y_0) \text{ is a maximum,}$$

and

$$f_{xx}(x_0, y_0)\, f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2 < 0 \Rightarrow (x_0, y_0) \text{ is a saddle point.}$$

For the sufficient conditions we need the second order partial derivatives of f. We have:

$$f_{xx}(x, y) = e^x + (x + y + a + 1)\, e^x = (x + y + a + 2)\, e^x,$$

$$\begin{aligned} f_{yy}(x,y) &= -e^y, \\ f_{xy}(x,y) &= e^x. \end{aligned}$$

It follows that for $x_0 = y_0 = -\frac{a+1}{2}$:

$$\begin{cases} f_{xx}(x_0,y_0) = e^{x_0} > 0 \\ f_{yy}(x_0,y_0) = -e^{y_0} = -e^{x_0} < 0 \\ f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2 = e^{x_0}\,(-e^{x_0}) - (e^{x_0})^2 = -2\,e^{2\,x_0} < 0 \end{cases}$$

Therefore, $P = \left(-\frac{a+1}{2}, -\frac{a+1}{2}\right)$ is a saddle point of f.

(b) (7 points)

We apply the Lagrange method. First of all we define the Lagrange function:

$$\begin{aligned} F(x,y,\lambda) &= f(x,y) + \lambda\,\varphi(x,y) \\ &= x^2 + y^2 + \lambda\,(a\,x^2 + b\,x\,y + 5\,y^2 - 16). \end{aligned}$$

The necessary conditions for constrained extreme points of f, given the constraint $\varphi(x,y) = 0$, are the so-called Lagrange conditions:

$$F_x(x,y,\lambda) = 0 \;\Rightarrow\; 2\,x + \lambda\,(2\,a\,x + b\,x) = 0, \tag{I}$$
$$F_y(x,y,\lambda) = 0 \;\Rightarrow\; 2\,y + \lambda\,(b\,x + 10\,y) = 0, \tag{II}$$
$$F_\lambda(x,y,\lambda) = 0 \;\Rightarrow\; a\,x^2 + b\,x\,y + 5\,y^2 - 16 = 0. \tag{III}$$

Therefore, in order that $(x,y) = (1,1)$ is a constrained extreme point of f, we must have:

$$F_x(1,1,\lambda) = 0 \;\Rightarrow\; 2 + \lambda\,(2\,a + b) = 0, \tag{IV}$$
$$F_y(1,1,\lambda) = 0 \;\Rightarrow\; 2 + \lambda\,(b + 10) = 0, \tag{V}$$
$$F_\lambda(1,1,\lambda) = 0 \;\Rightarrow\; a + b - 11 = 0. \tag{VI}$$

From (IV) we obtain

$$\lambda = -\frac{2}{2\,a + b}. \tag{VII}$$

From (V) we obtain

$$\lambda = -\frac{2}{b + 10}. \tag{VIII}$$

Equations (VII) and (VIII) imply that:

$$-\frac{2}{2\,a + b} = \lambda = -\frac{2}{b + 10} \Leftrightarrow 2\,a + b = b + 10 \Leftrightarrow 2\,a = 10 \Leftrightarrow a = 5.$$

We plug this result into (VI) and we have:

$$5 + b - 11 = 0 \Leftrightarrow b = 6.$$

(c) (5 points)

To solve the integral, we apply the substitution method. Let

$$u = \cos(x^2).$$

It follows that

$$\frac{du}{dx} = -2\,x\,\sin(x^2) \quad \Leftrightarrow \quad -\frac{1}{2}\,du = x\,\sin(x^2)\,dx.$$

We obtain:

$$\int x\,\sin(x^2)\,(\cos(x^2))^3\,dx = \int -\frac{1}{2}\,u^3\,du = -\frac{1}{8}\,u^4 + C = -\frac{1}{8}\,(\cos(x^2))^4 + C, \quad C \in \mathbb{R}.$$

It follows that:

$$\int_0^{\sqrt{0.5\,\pi}} x\,\sin(x^2)\,(\cos(x^2))^3\,dx = \left[-\frac{1}{8}\,(\cos(x^2))^4\right]_0^{\sqrt{0.5\,\pi}} = 0 - \left(-\frac{1}{8}\right) = \frac{1}{8}.$$

(d) (6 points)

We have:

$$\int_0^e |\ln(x)|\;dx = \int_0^1 |\ln(x)|\;dx + \int_1^e |\ln(x)|\;dx \overset{\ln(x)<0 \Leftrightarrow 0<x<1}{=} \underbrace{\int_0^1 (-\ln(x))\,dx}_{\text{improper integral}} + \int_1^e \ln(x)\,dx.$$

We first solve the indefinite integral $\int \ln(x)\,dx$ using integration by parts. Let $u(x) = \ln(x)$ and $v'(x) = 1$, then:

$$\int \ln(x)\,dx = \int \underbrace{1}_{=v'(x)} \cdot \underbrace{\ln(x)}_{=u(x)}\,dx = \underbrace{x}_{=v(x)}\,\underbrace{\ln(x)}_{=u(x)} - \int \underbrace{x}_{=v(x)}\,\underbrace{\frac{1}{x}}_{=u'(x)}\,dx = x\,\ln(x) - \int 1\,dx = x\,\ln(x) - x + C.$$

We obtain:

$$\int_1^e \ln(x)\,dx = [x\,(\ln(x)-1)]_1^e = 0 - (-1) = 1$$

and

$$\int_0^1 (-\ln(x))\,dx = \lim_{a \searrow 0} \int_a^1 (-\ln(x))\,dx = -\lim_{a \searrow 0}\,[x\,(\ln(x)-1)]_a^1 = 1 + \lim_{a \searrow 0}(a\,\ln(a) - a) = 1.$$

Adding up the two integrals, we finally have:

$$\int_0^e |\ln(x)|\;dx = 2.$$

Exercise 2

(a) (4 points)

The following properties hold:

(i) $(A\,B)^T = B^T\,A^T$;

(ii) $(A\,B)^{-1} = B^{-1}\,A^{-1}$;

(iii) $(A^T)^{-1} = (A^{-1})^T$;

(iv) $A^{-1}\,A = I$;

(v) $A = A^T$ if and only if A is symmetric.

We obtain:

$$\begin{aligned}
B^T\,(A\,B)^T\,(B^{-1}\,A^{-1})^T\,B\,(A\,B)^{-1} &= B^T(B^T\,A^T)\,((A^{-1})^T\,(B^{-1})^T)\,B\,(B^{-1}\,A^{-1}) \\
&= B^T\,B^T\,\underbrace{A^T\,(A^T)^{-1}}_{=I}\,(B^T)^{-1}\,\underbrace{B\,B^{-1}}_{=I}\,A^{-1} \\
&= B^T\,\underbrace{B^T\,(B^T)^{-1}}_{=I}\,A^{-1} \\
&= B^T\,A^{-1} \\
&\overset{A \text{ is symmetric}}{=} B^T\,(A^T)^{-1} \\
&= B^T\,(A^{-1})^T \\
&= (A^{-1}\,B)^T.
\end{aligned}$$

Therefore we proved that:

$$B^T\,(A\,B)^T\,(B^{-1}\,A^{-1})^T\,B\,(A\,B)^{-1} = (A^{-1}\,B)^T.$$

(b) (4 points)

The direction of the steepest ascent of f at the point $(x_0, y_0) = (8, 2)$ is given by the vector

$$\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

if and only if

$$\lambda\,\mathbf{b} = \mathbf{grad} f(8, 2)$$

for some $\lambda > 0$.

We have:

$$\mathbf{grad} f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix} = \begin{pmatrix} \frac{a}{x-2} + y^2 \\ 2\,x\,y + 8 \end{pmatrix}.$$

It follows that:

$$\mathbf{grad}f(8,2) = \begin{pmatrix} \frac{1}{6}\,a + 4 \\ 40 \end{pmatrix}.$$

Therefore, we have:

$$\begin{aligned} &\lambda\,\mathbf{b} = \mathbf{grad}f(8,2) \\ \Leftrightarrow\quad &\lambda \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{6}\,a + 4 \\ 40 \end{pmatrix}. \end{aligned}$$

It follows that $\lambda = 10$ and

$$\frac{1}{6}\,a + 4 = 30 \Leftrightarrow a = 156.$$

(c) (3 points)
The following holds: the system of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is no basis of the 3-dimensional space $\mathbb{R}^3$ if and only if $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly dependent, i.e., the matrix $[\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ is singular, which is equivalent to $\det([\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]) = 0$.
We have:

$$\begin{aligned} \det([\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]) &= \left| \begin{pmatrix} 1 & 2\,t & 8 \\ t & 4 & t \\ 0 & t & t^2 \end{pmatrix} \right| \\ &= 4\,t^2 + 8\,t^2 - t^2 - 2\,t^4 \\ &= 11\,t^2 - 2\,t^4 \\ &= t^2\,(11 - 2\,t^2). \end{aligned}$$

Therefore,

$$\det([\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]) = 0 \Leftrightarrow t^2\,(11 - 2\,t^2) = 0 \Leftrightarrow t \in \left\{ -\sqrt{\frac{11}{2}}, 0, \sqrt{\frac{11}{2}} \right\}.$$

It follows that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is no basis of the 3-dimensional space $\mathbb{R}^3$ if and only if $t \in \left\{0, \pm\sqrt{\frac{11}{2}}\right\}$.

(d) (6 points)

λ is an eigenvalue of M if and only if $\det(M - \lambda\,I) = 0$. We have:

$$\det(M - \lambda\,I) = \left| \begin{pmatrix} -\lambda & 2\,a \\ -3\,a & 5\,a - \lambda \end{pmatrix} \right| = \lambda^2 - 5\,a\,\lambda + 6\,a^2 = (\lambda - 2\,a)\,(\lambda - 3\,a)$$

and thus

$$\det(M - \lambda\,I) = 0 \Leftrightarrow \lambda \in \{2\,a, 3\,a\}.$$

We now compute the eigenvectors. The vector $\mathbf{x} = (x, y)^T$ is an eigenvector of M associated with the eigenvalue λ if and only if $(M - \lambda\, I)\,\mathbf{x} = \mathbf{0}$.

For $\lambda = \lambda_1 = 2\,a$ we have:

$$\begin{pmatrix} -2\,a & 2\,a \\ -3\,a & 3\,a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} -2\,a\,x + 2\,a\,y &= 0 \\ -3\,a\,x + 3\,a\,y &= 0 \end{cases} \Leftrightarrow x = y,$$

i.e.,

$$\mathbf{x}_1 = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R} \setminus \{0\}$$

are the eigenvectors of M associated with the eigenvalue $\lambda_1 = 2\,a$.

For $\lambda = \lambda_2 = 3\,a$ we have:

$$\begin{pmatrix} -3\,a & 2\,a \\ -3\,a & 2\,a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow -3\,a\,x + 2\,a\,y = 0 \Leftrightarrow 3\,x = 2\,y,$$

i.e.,

$$\mathbf{x}_2 = s \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

are the eigenvectors of M associated with the eigenvalue $\lambda_2 = 3\,a$.

Finally, because the vector $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ corresponds to $\mathbf{x}_1$ when $t = 2$, and thus is an eigenvector of M assocciated with the eigenvalue $\lambda_1 = 2\,a$, we have:

$$M^n\,\mathbf{x} = (2\,a)^n \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2^{n+1}\,a^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(e) (8 points)

We apply the Gauss method:

$$(A, \mathbf{b}) \quad = \quad \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & 5 & -1 & 0 \\ 1 & 2 & 4 & -1 & 2 & 0 \end{array}\right) \end{array} \begin{array}{l} \\ -2\,(I) \\ -(I) \end{array}$$

$$\rightarrow \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \end{array}\right) \end{array} \begin{array}{l} -(II) \\ \\ -(II) \end{array}$$

$$\rightarrow \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -2 & 4 & 0 \\ 0 & 1 & 2 & 3 & -3 & 0 \\ 0 & 0 & 1 & -5 & 4 & 0 \end{array}\right) \end{array} \begin{array}{l} +(III) \\ -2\,(III) \\ \end{array}$$

$$(A^\star, b^\star) = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -7 & 8 & 0 \\ 0 & 1 & 0 & 13 & -11 & 0 \\ 0 & 0 & 1 & -5 & 4 & 0 \end{array}\right) \end{array} .$$

It follows that:

$$\begin{aligned} x_1 &= 7\,x_4 - 8\,x_5 \\ x_2 &= -13\,x_4 + 11\,x_5 \\ x_3 &= 5\,x_4 - 4\,x_5 \end{aligned}$$

and $x_4 = t$, $x_5 = s$ are free variables. Therefore, the solution set corresponds to:

$$W = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t \begin{pmatrix} 7 \\ -13 \\ 5 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -8 \\ 11 \\ -4 \\ 0 \\ 1 \end{pmatrix} ; \quad t, s \in \mathbb{R} \right\} .$$

A basis of W is:

$$\left\{ \begin{pmatrix} 7 \\ -13 \\ 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ 11 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\} .$$

Part II: Multiple-choice Questions

Exercise 3

	(a)	(b)	(c)	(d)
Question 1	☐	☐	☐	☒
Question 2	☐	☒	☐	☐
Question 3	☐	☐	☒	☐
Question 4	☐	☒	☐	☐
Question 5	☐	☐	☒	☐
Question 6	☒	☐	☐	☐
Question 7	☐	☒	☐	☐
Question 8	☒	☐	☐	☐

Q1. (d). The constraint $\varphi(x,y) = \frac{x^2}{25} + \frac{y^2}{9} = 1$ corresponds to an ellipse with centre $(0,0)$ and semi-axes 5 and 3. The point on the ellipse with the lowest y-value is $P = (0,-3)$.

Q2. (b). A function f is a density function on $I \subseteq \mathbb{R}$ if and only if (i) $f(x) \geq 0$ for all $x \in I$, and (ii) $\int_I f(x)\, dx = 1$. Because,

$$\int_0^4 f(x)\, dx = \int_0^4 \left(a\, x + \frac{1}{8}\right)\, dx = \left[\frac{a}{2}\, x^2 + \frac{1}{8}\, x\right]_0^4 = 8\, a + \frac{1}{2}$$

then

$$\int_0^4 f(x)\, dx = 1 \Leftrightarrow 8\, a + \frac{1}{2} = 1 \Leftrightarrow a = \frac{1}{16}.$$

Moreover, for $a = \frac{1}{16}$

$$f(x) = a\, x + \frac{1}{8} = \frac{1}{16}\, x + \frac{1}{8} \geq \frac{1}{8} \geq 0$$

for $x \in [0,4]$.

Q3. (c). Only (c) is correct, because when the function f is differentiable on $[a,b]$, then it is also continuous on $[a,b]$, and thus the definite integral of f on $[a,b]$ exists. However, when the definite integral of f on $[a,b]$ exists, then f does not have to be continuous on $[a,b]$.

Q4. (b). Because

$$\det(C) = \det(A^{-1})\ \det(B)\ \det(A) = \frac{1}{\det(A)}\ \det(B)\ \det(A) = \det(B) = 2,$$

then

$$\det(C^n) = (\det(C))^n = 2^n.$$

Q5. (c). Because $\det([\mathbf{a},\mathbf{b},\mathbf{c}]) = -6 \neq 0$, then the system of vectors $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is linearly independent, thus a basis of $\mathbb{R}^3$. It follows that $\mathbf{d}$ is a linear combination of $\mathbf{a},\mathbf{b},\mathbf{c}$ for all $t \in \mathbb{R}$.

Q6. (a). Because the system has at least on solution, then $\text{rg}(A) = \text{rg}(A,\mathbf{b})$. Moreover, because the dimension of the solution space is $2 = 5 - \text{rg}(A)$, then $3 = \text{rg}(A) = \text{rg}(A,\mathbf{b})$.

Q7. (b). First of all, $f(x) = \ln(x\,e^x) = \ln(x) + \ln(e^x) = \ln(x) + x$. Second,

$$\left(x\,\ln(x) + \frac{x^2}{2} - x\right)' = \ln(x) + x\,\frac{1}{x} + x - 1 = \ln(x) + x = f(x).$$

Therefore,

$$\int \ln(x\,e^x)\,dx = x\,\ln(x) + \frac{x^2}{2} - x + C.$$

It can be easily shown that the functions in answers (a) and (c) are *no* antiderivatives of f.

Q8. (a). λ is an eigenvalue of A if and only if $\det(A - \lambda\,I) = 0$. We have:

$$0 = \det(A - \lambda\,I) = \left|\begin{pmatrix} 2-\lambda & a \\ a & 2-\lambda \end{pmatrix}\right| = (2-\lambda)^2 - a^2 \Leftrightarrow 2 - \lambda = \pm|a| \Leftrightarrow \lambda = 2 \pm |a|.$$

Therefore, A has two distinct real-valued eigenvalues if and only if $a \neq 0$.

Exercise 4

	(a)	(b)	(c)	(d)
Question 1	☒	☐	☐	☐
Question 2	☐	☐	☒	☐
Question 3	☒	☐	☐	☐
Question 4	☐	☒	☐	☐
Question 5	☐	☐	☒	☐
Question 6	☐	☒	☐	☐
Question 7	☐	☐	☒	☐
Question 8	☒	☐	☐	☐

Q1. (a). Because $2\,\sin(x)\,\cos(x) = \sin(2\,x)$, then

$$\int_0^{\pi} 2\,\sin(x)\,\cos(x)\,dx = \int_0^{\pi} \sin(2\,x)\,dx.$$

Moreover, because the graph of $\sin(2\,x)$ is symmetric around the x-axes, then:

$$\int_0^{\pi} \sin(2\,x)\,dx = 0.$$

Alternatively, one can use integration by substitution with $u = \sin(x)$ to obtain:

$$\int_0^{\pi} 2\,\sin(x)\,\cos(x)\,dx = \int_0^{0} 2\,u\,du = \left[u^2\right]_0^0 = \left[(\sin(x))^2\right]_0^0 = 0.$$

Q2. (c). $\mathbf{u}$ is orthogonal to $\mathbf{v}$ if and only if

$$\begin{aligned} &\mathbf{u}\cdot\mathbf{v} = 0 \\ &\Leftrightarrow\quad (t-2) + t\,(t-1) + 27 = 0 \\ &\Leftrightarrow\quad t^2 + 25 = 0. \end{aligned}$$

This latter equation is never satisfied on $\mathbb{R}$. Therefore, $\mathbf{u}$ is never orthogonal to $\mathbf{v}$.

Q3. (a). We apply the Gauss method:

$$A = \begin{pmatrix} 1 & 1 & 3 & -1 & -2 \\ 3 & -5 & -7 & 13 & -10 \\ -1 & 3 & 5 & -7 & 4 \\ -2 & 10 & 18 & -22 & 10 \end{pmatrix} \begin{matrix} \\ -3\,(I) \\ +(I) \\ +2\,(I) \end{matrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 & -1 & -2 \\ 0 & -8 & -16 & 16 & -4 \\ 0 & 4 & 8 & -8 & 2 \\ 0 & 12 & 24 & -24 & 6 \end{pmatrix} \begin{matrix} \\ :(-8) \\ \\ \\ \end{matrix}$$

$$\to \begin{pmatrix} 1 & 1 & 3 & -1 & -2 \\ 0 & 1 & 2 & -2 & 0.5 \\ 0 & 4 & 8 & -8 & 2 \\ 0 & 12 & 24 & -24 & 6 \end{pmatrix} \begin{matrix} -(II) \\ \\ -4\,(II) \\ -12\,(II) \end{matrix}$$

$$A^\star = \begin{pmatrix} 1 & 0 & 1 & 1 & -2.5 \\ 0 & 1 & 2 & -2 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $\mathrm{rg}(A^\star) = 2$ and thus $\mathrm{rg}(A) = \mathrm{rg}(A^\star) = 2$.

Q4. (b). Because

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

is invertible (its determinant is equal to 1, and thus differs from 0) we have:

$$X = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}.$$

Q5. (c). Let λ_i be an eigenvalue of A associated with eigenvector $\mathbf{x}_i$. It follows that:

$$A^2\,\mathbf{x}_i = A\,(A\,\mathbf{x}_i) = A(\lambda_i\,\mathbf{x}_i) = \lambda_i\,A\,\mathbf{x}_i = \lambda_i\,(\lambda_i\,\mathbf{x}_i) = \lambda_i^2\,\mathbf{x}_i.$$

Therefore, λ_i^2 is an eigenvalue of A^2 associated with eigenvector $\mathbf{x}_i$.

By contrast, λ_i and $2\,\lambda_i$ are generally no eigenvalues of A^2. Take $A = 3\,I$, i.e., A is a diagonal matrix with all diagonal elements equal 3. The unique eigenvalue of A is $\lambda = 3$. Because $A^2 = 9\,I$, the unique eigenvalue of A^2 is $9 = 3^2 = \lambda^2$.

Q6. (b). Answer (c) and (d) do not satisfy the condition $y_0 = 2$. For answer (a) we have:

$$y_{k+1} - (1+a)\,y_k = 2\,(1+a)^{k+1} - (1+a)\,2\,(1+a)^k = 0 \neq a.$$

By contrast, answer (b) satisfies $y_0 = 2$ and

$$y_{k+1} - (1+a)\,y_k = 2\,(1+a)^{k+1} - 1 - (1+a)\,(3\,(1+a)^k - 1) = 3\,(1+a)^{k+1} - 1 - 3\,(1+a)^{k+1} + 1 + a = a.$$

Alternatively, one can solve the initial value problem (normal form)

$$y_{k+1} = \underbrace{(1+a)}_{=A}\,y_k + \underbrace{a}_{=B}, \qquad y_0 = 2$$

to obtain

$$y^\star = \frac{B}{1-A} = \frac{a}{1-(1+a)} = -1$$

and

$$y_{k+1} = A^k\,(y_0 - y^\star) + y^\star = (1+a)^k\,(2+1) + (-1) = 3\,(1+a)^k - 1,$$

which correspond to the sequence in answer (b).

Q7. (c). The normal form of the difference equation is

$$y_{k+1} = \frac{1}{3}\, y_k + 5.$$

It follows that $A = \frac{1}{3}$ and $B = 5$. Because $A > 0$ and $|A| < 1$, then the general solution of the difference equation is monotone and convergent.

Q8. (a). The normal form of the difference equation is

$$y_{k+1} = \frac{c-1}{c+2}\, y_k + \frac{5}{c+2},$$

i.e., $A = \frac{c-1}{c+2}$ and $B = \frac{5}{c+2}$. Because $B \neq 0$, the general solution of the difference equation is monotone and divergent if and only if $A > 1$. We have:

Case 1: $c + 2 > 0 \Leftrightarrow c > -2$:

$$A > 1 \Leftrightarrow \frac{c-1}{c+2} > 1 \Leftrightarrow c - 1 > c + 2 \Leftrightarrow -1 > 2 \ \rightarrow \ \text{impossible.}$$

Case 2: $c + 2 < 0 \Leftrightarrow c < -2$:

$$A > 1 \Leftrightarrow \frac{c-1}{c+2} > 1 \Leftrightarrow c - 1 < c + 2 \Leftrightarrow -1 < 2 \ \rightarrow \ \text{always true.}$$

It follows that the general solution of the difference equation is monotone and divergent if and only if $c < -2$.