Mathematics B
Exam Spring Semester 2017

Dr. Reto Schuppli*

26 June 2017

Part I: Open Questions (50 points)

General instructions for the open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain a single version.

(vi) Temporary computations or sketches must be reported in separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (25 points)

(a) (7 points)

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of two variables defined by

$$f(x, y) = (x + y + a)\, e^{x} - e^{y}, \quad \text{where } a \in \mathbb{R}.$$

Examine the function f for *stationary points*, i.e., maxima, minima, and saddle points.

Exercise 1

(a) (additional space for your solution)

Exercise 1

(b) (7 points)

The function

$$f(x,y) = x^2 + y^2$$

has to be optimised under the constraint

$$\varphi(x,y) = a\,x^2 + b\,x\,y + 5\,y^2 - 16 = 0.$$

Determine the parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that $(1,1)$ is a possible extreme point.

Remark:

It is not required to verify if $(1,1)$ is indeed an extreme point, and which type (maximum or minimum) of extreme point.

Exercise 1

(b) (additional space for your solution)

Exercise 1

(c) (5 points)

Compute

$$\int_0^{\sqrt{0.5\,\pi}} x\,\sin(x^2)\,\left(\cos(x^2)\right)^3\,dx.$$

Exercise 1

(c) (additional space for your solution)

Exercise 1

(d) (6 points)

Compute

$$\int_0^e |\ln(x)|\, dx.$$

Remark: You can apply the result proved in Mathematics A that:

$$\lim_{x \searrow 0} x \ln(x) = 0.$$

Exercise 1

(d) (additional space for your solution)

Exercise 2 (25 points)

(a) (4 points)

The square $n \times n$ matrices A and B are regular, and A is also symmetric.

Prove that:

$$B^T\,(AB)^T\,(B^{-1}A^{-1})^T\,B\,(AB)^{-1} = (A^{-1}B)^T.$$

Exercise 2

(b) (4 points)

Given is the function

$$f(x,y) = a\,\ln(x-2) + x\,y^2 + 8\,y, \quad \text{where } a \in \mathbb{R}.$$

Compute the gradient of f at $(x_0, y_0) = (8, 2)$.

How must the parameter $a \in \mathbb{R}$ be chosen, so that the direction of the steepest ascent of the function f at the point $(x_0, y_0) = (8, 2)$ is given by the vector $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

Exercise 2

(b) (additional space for your solution)

Exercise 2

(c) (3 points)

Given are the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ t \\ 0 \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} 2t \\ 4 \\ t \end{pmatrix}, \ \mathbf{b}_3 = \begin{pmatrix} 8 \\ t \\ t^2 \end{pmatrix}.$$

For which values of $t \in \mathbb{R}$ is the system of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ *no* basis of the 3-dimensional space $\mathbb{R}^3$?

Exercise 2

(d) (6 points)

Given is the matrix

$$M = \begin{pmatrix} 0 & 2a \\ -3a & 5a \end{pmatrix}, \text{ where } a \neq 0.$$

Compute the eigenvalues and the eigenvectors of matrix M.

Compute the vector $M^n \mathbf{x}$, where $\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

Exercise 2

(d) (additional space for your solution)

Exercise 2

(e) (8 points)

Given is the system of linear equations

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 0 \\ 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & - & x_5 & = & 0 \\ x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & + & 2x_5 & = & 0 \end{array}.$$

Compute the general solution of the system using the Gaussian elimination method.

Give a basis of the vector space

$W = \{\mathbf{x} \in \mathbb{R}^5 \,|\, x_1+x_2+x_3+x_4+x_5 = 0 \wedge 2x_1+3x_2+4x_3+5x_4-x_5 = 0 \wedge x_1+2x_2+4x_3-x_4+2x_5 = 0\}$.

Exercise 2

(d) (additional space for your solution)

Part II: Multiple-choice Question (50 points)

General instructions for multiple-choice questions:

(i) The solution must be reported in the multiple-choice solution form, which is distributed together with the exam. Only the answers reported in the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility can be marked.

(iii) When two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions.

Exercise 3 (25 points)

Question 1 (3 points)

The function $f(x,y) = y$ has under constraint $\varphi(x,y) = \frac{x^2}{25} + \frac{y^2}{9} = 1$ its minimum at

(a) $P = (-5, 0)$.

(b) $P = (0, 3)$.

(c) $P = (0, 0)$.

(d) $P = (0, -3)$.

Exercise 3

Question 2 (4 points)

Given is the function

$$f(x) = \begin{cases} ax + \frac{1}{8} & \text{for } 0 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases} .$$

(a) f is a density function for all $a \in \mathbb{R}$.

(b) f is a density function only for $a = \frac{1}{16}$.

(c) f is a density function only for $a = -\frac{1}{16}$.

(d) f is a density function for no $a \in \mathbb{R}$.

Exercise 3

Question 3 (2 points)

Let $f : [a, b] \to \mathbb{R}$ be any function defined on the interval $[a, b]$.

Which of the following statements is *correct*?

(a) If the definite integral of f on $[a, b]$ exists, then f is continuous on $[a, b]$.

(b) If f is not continuous on $[a, b]$, then the definite integral of f on $[a, b]$ does not exist.

(c) If f is differentiable on $[a, b]$, then the definite integral of f on $[a, b]$ exists.

(d) The definite integral of f on $[a, b]$ always exists.

Exercise 3

Question 4 (2 points)

A and B are square matrices with $\det(A) = 5$ and $\det(B) = 2$; the matrix C is defined by $C = A^{-1} B A$.

(a) For all $n \in \mathbb{N}$ we have: $\det(C^n) = 1$.

(b) For all $n \in \mathbb{N}$ we have: $\det(C^n) = 2^n$.

(c) For all $n \in \mathbb{N}$ we have: $\det(C^n) = 2^n \cdot 5^n$.

(d) None of the above answers are correct.

Exercise 3

Question 5 (4 points)

Given are the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ t \end{pmatrix}.$$

(a) Only for $t = 6$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) Only for $t = 6$ and $t = 0$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(c) For all $t \in \mathbb{R}$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(d) For no $t \in \mathbb{R}$ it is possible to write $\mathbf{d}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Exercise 3

Question 6 (2 points)

A is a 6×5 matrix. The system of linear equations $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions and the solution space has dimension 2. We have:

(a) $rg(A) = rg(A; \mathbf{b}) = 3$.

(b) $rg(A) = rg(A; \mathbf{b}) = 4$.

(c) $rg(A) < rg(A; \mathbf{b}) = 4$.

(d) None of the above answers is correct.

Exercise 3

Question 7 (4 points)

The indefinite integral

$$\int \ln(x\, e^x)\, dx\,,\ (x > 0)$$

is

(a) $x\ \ln(x) + x^2 - x + C$.

(b) $x\ \ln(x) + \frac{x^2}{2} - x + C$.

(c) $x\ \ln(x) + x^2 + C$.

(d) None of the above answers is correct.

Exercise 3

Question 8 (4 points)

Given is the matrix

$$A = \begin{pmatrix} 2 & a \\ a & 2 \end{pmatrix}, \text{ where } a \neq 0.$$

(a) The matrix A has two distinct real eigenvalues for all $a \neq 0$ in $\mathbb{R}$.

(b) The matrix A has exactly one real eigenvalue for all $a \neq 0$ in $\mathbb{R}$.

(c) The matrix A has no real eigenvalue for all $a \neq 0$ in $\mathbb{R}$.

(d) Depending on $a \neq 0$, the matrix A has no, one or two real eigenvalues.

Exercise 4 (25 points)

Question 1 (3 points)

The definite integral

$$\int_0^\pi 2\,\sin(x)\,\cos(x)\,dx$$

has the value

(a) 0.

(b) 1.

(c) 2.

(d) None of the above results is correct.

Exercise 4

Question 2 (3 points)

For which value of $t \in \mathbb{R}$ are the vectors $\mathbf{u} = \begin{pmatrix} t-2 \\ t \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ t-1 \\ 9 \end{pmatrix}$ orthogonal?

(a) $t = 5$.

(b) $t = 5$ or $t = -5$.

(c) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal for no $t \in \mathbb{R}$.

(d) $\mathbf{u}$ and $\mathbf{v}$ are orthogonal for all $t \in \mathbb{R}$.

Exercise 4

Question 3 (4 points)

The 4×5 matrix

$$A = \begin{pmatrix} 1 & 1 & 3 & -1 & -2 \\ 3 & -5 & -7 & 13 & -10 \\ -1 & 3 & 5 & -7 & 4 \\ -2 & 10 & 18 & -22 & 10 \end{pmatrix}$$

(a) has rank 2.

(b) has rank 3.

(c) has rank 4.

(d) has rank 5.

Exercise 4

Question 4 (4 points)

We search for a matrix X, such that

$$X \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}.$$

(a) $X = \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}.$

(b) $X = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}.$

(c) $X = \begin{pmatrix} 4 & -3 \\ -2 & 4 \end{pmatrix}.$

(d) There is no matrix X, that satisfies the equation above.

Exercise 4

Question 5 (2 points)

The $n \times n$ matrix A has the eigenvalues λ_1, λ_2, ..., λ_n. It follows that the matrix A^2

(a) has the same eigenvalues.

(b) has eigenvalues $2\lambda_1$, $2\lambda_2$, ..., $2\lambda_n$.

(c) has eigenvalues λ_1^2, λ_2^2, ..., λ_n^2.

(d) None of the above answers is correct.

Exercise 4

Question 6 (3 points)

The initial value problem

$$y_{k+1} - (1+a)\,y_k = a, \text{ where } a \neq -1,\ a \neq 0,$$
$$y_0 = 2$$

has the solution

(a) $y_k = 2\,(1+a)^k$.

(b) $y_k = 3\,(1+a)^k - 1$.

(c) $y_k = 4\,(1+a)^k - 1$.

(d) $y_k = 5\,(1+a)^k - 2$.

Exercise 4

Question 7 (2 points)

The general solution of the difference equation

$$3\,(y_k - y_{k+1}) + 3 = 2\,y_k - 12$$

is

(a) oscillating and convergent.

(b) oscillating and divergent.

(c) monotone and convergent.

(d) monotone and divergent.

Exercise 4

Question 8 (4 points)

The general solution of the difference equation

$$(2+c)\,y_{k+1} + (1-c)\,y_k = 5,$$

where $c \in \mathbb{R} \setminus \{-2\}$ is monotone and divergent if and only if

(a) $c < -2$.

(b) $c \in (-2, 0)$.

(c) $c < -\frac{1}{2}$.

(d) The general solution of the difference equation is for no $c \in \mathbb{R} \setminus \{-2\}$ monotone and divergent.

Exams Assessment Level: Spring Semester 2017

2'202 Mathematics B

Multiple-choice answer sheet

Exercise 3 (25 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐

Exams Assessment Level: Spring Semester 2017

2'202 Mathematics B

Exercise 4 (25 points)

Question 1: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
1.	☐	☐	☐	☐

Question 2: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
2.	☐	☐	☐	☐

Question 3: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
3.	☐	☐	☐	☐

Question 4: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
4.	☐	☐	☐	☐

Question 5: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
5.	☐	☐	☐	☐

Question 6: Single-Choice (3 points)

	(a)	(b)	(c)	(d)
6.	☐	☐	☐	☐

Question 7: Single-Choice (2 points)

	(a)	(b)	(c)	(d)
7.	☐	☐	☐	☐

Question 8: Single-Choice (4 points)

	(a)	(b)	(c)	(d)
8.	☐	☐	☐	☐