Mathematics A
Master Solutions Alternative Date Exam
Autumn Semester 2016

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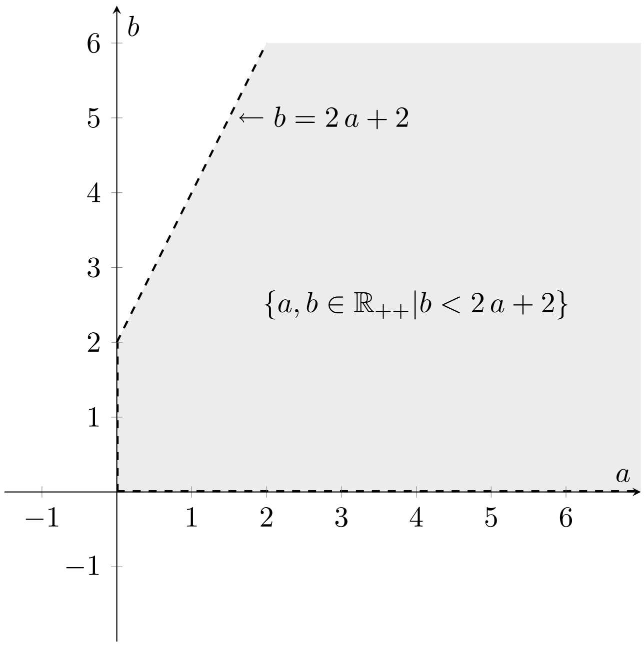
Part I: Open questions

Exercise 1

(a) (**6 points**).

The series $\sum_{t=0}^{\infty}\left(\frac{1-b}{1+2\,a}\right)^t$ is a geometric series with a $q = \frac{1-b}{1+2\,a}$. Therefore, it converges if and only if $|q| < 1$. It follows that:

$$\begin{aligned}
|q| < 1 \quad &\Leftrightarrow \quad \left|\frac{1-b}{1+2\,a}\right| < 1 \\
&\Leftrightarrow \quad -1 < \frac{1-b}{1+2\,a} < 1 \\
&\overset{1+2\,a>0}{\Leftrightarrow} \quad -1-2\,a < \underbrace{1-b < 1+2\,a}_{\text{true for } a,b\in\mathbb{R}_{++}} \\
&\overset{a,b\in\mathbb{R}_{++}}{\Leftrightarrow} \quad 0 < b < 2\,a+2.
\end{aligned}$$



If $b < 2\,a+2$ and $a, b \in \mathbb{R}_{++}$, then it follows:

$$\sum_{t=0}^{\infty}\left(\frac{1-b}{1+2\,a}\right)^t = \frac{1}{1-q} = \frac{1}{1-\frac{1-b}{2\,a+1}} = \frac{2\,a+1}{2\,a+b}.$$

(b) (**6 points**).
The constant payments $C^I = 30,000$ CHF at the end of each year for $n = 16$ years correspond to a 16-year annuity immediate. Therefore, the value of this annuity after 16 years is:

$$\begin{aligned} A^I_{16} &= C^I \, \frac{(1+i)^n - 1}{i} \\ &= 30,000 \, \frac{1.05^{16} - 1}{0.05} \\ &\approx 709,724.75 \text{ (CHF)}, \end{aligned}$$

and its present value corresponds to:

$$P^I_{10} = \frac{A^I_{16}}{(1+0.05)^{16}} = \frac{30,000}{0.05} \, \frac{1.05^{16} - 1}{1.05^{16}} \approx 325,133.10 \text{ (CHF)}.$$

Similarly, the constant payments B at the beginning of each year for 10 years correspond to a 10-year annuity due. Its present value is:

$$P^D_{10} = \frac{B}{0.05} \cdot 1.05 \cdot \frac{1.05^{10} - 1}{1.05^{10}}.$$

Because $P^I_{16} = P^D_{10}$, we have:

$$\frac{30,000}{0.05} \, \frac{1.05^{16} - 1}{1.05^{16}} = \frac{B}{0.05} \cdot 1.05 \cdot \frac{1.05^{10} - 1}{1.05^{10}}.$$

It follows:

$$B = \frac{30,000}{1.05^7} \, \frac{1.05^{16} - 1}{1.05^{10} - 1} \approx 40,101.15 \text{ (CHF)}.$$

(c) (**4 points**).

We have:

$$\begin{aligned} &\lim_{t \to 2} \frac{\sqrt{t+2} - \sqrt{2\,t}}{\sqrt{t+4} - \sqrt{3\,t}} \\ &\overset{\text{de l'Hôpital}}{=} \lim_{t \to 2} \frac{\frac{1}{2\sqrt{t+2}} - \frac{1}{\sqrt{2\,t}}}{\frac{1}{2\sqrt{t+4}} - \frac{3}{2\sqrt{3\,t}}} \\ &= \frac{\frac{1}{4} - \frac{1}{2}}{\frac{1}{2\sqrt{6}} - \frac{3}{2\sqrt{6}}} \\ &= \frac{-\frac{1}{4}}{-\frac{2}{2\sqrt{6}}} \\ &= \frac{\sqrt{6}}{4}. \end{aligned}$$

(d1) (**4 points**).

$$x \in D_f \quad \Leftrightarrow \quad \begin{cases} \sqrt{4-x} > 0 & \text{(because of the } \ln(\cdot)\text{)} \\ 4-x \geq 0 & \text{(because of the } \sqrt{(\cdot)}\text{)} \end{cases}$$
$$\Leftrightarrow \quad 4-x > 0$$
$$\Leftrightarrow \quad x < 4.$$

Therefore,

$$D_f = (-\infty, 4).$$

Moreover,

$$\begin{aligned} -\infty < x < 4 \quad &\Leftrightarrow \quad 0 < \sqrt{4-x} < \infty \\ &\Leftrightarrow \quad -\infty < \ln(\sqrt{4-x}) < \infty \\ &\Leftrightarrow \quad -\infty < 1 - \ln(\sqrt{4-x}) < \infty \\ &\Leftrightarrow \quad 0 < e^{1-\ln(\sqrt{4-x})} < \infty. \end{aligned}$$

It follows that:

$$R_f = \mathbb{R}_{++}.$$

(d2) (**3 points**).
Because f is a differentiable function on its domain, we can use the first derivative of f to study the monotonicity properties of the function f. We have:

$$f'(x) = \underbrace{e^{1-\ln(\sqrt{4-x})}}_{>0} \underbrace{\left(-\frac{1}{\sqrt{4-x}}\right)}_{<0} \underbrace{\frac{1}{2\sqrt{4-x}}}_{>0} \underbrace{(-1)}_{<0} > 0.$$

If follows that $f'(x) > 0$ for all $x \in D_f$ and thus f is strictly monotonically increasing on its domain.

(d3) (**3 points**).

For $y \in R_f$ we have:

$$\begin{aligned} y = e^{1-\ln(\sqrt{4-x})} = e^1 \cdot e^{-\ln(\sqrt{4-x})} &= \frac{e}{\sqrt{4-x}} \\ \Leftrightarrow \quad \frac{e}{y} &= \sqrt{4-x} \\ \Leftrightarrow \quad 4 - x &= \frac{e^2}{y^2} \\ \Leftrightarrow \quad x &= 4 - \frac{e^2}{y^2}. \end{aligned}$$

Because $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$, then it follows that:

$$f^{-1} : \mathbb{R} \to (-\infty, 4), \quad x \mapsto y = f^{-1}(x) = 4 - \frac{e^2}{x^2}.$$

Exercise 2

(a1) (**4 points**).
The third order Taylor polynomial of f at x_0 is given by:

$$P_3(x) = f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2}\,(x - x_0)^2 + \frac{f'''(x_0)}{3!}\,(x - x_0)^3.$$

With $x_0 = 0$ we have:

$$\begin{aligned}
f(x_0) &= f(0) = (1+0)^{\frac{1}{4}} = 1 \\
f'(x) &= \frac{1}{4}\,(1+x)^{-\frac{3}{4}} \Rightarrow f'(x_0) = f'(0) = \frac{1}{4} \\
f''(x) &= -\frac{3}{16}\,(1+x)^{-\frac{7}{4}} \Rightarrow f''(x_0) = f''(0) = -\frac{3}{16} \\
f'''(x) &= \frac{21}{64}\,(1+x)^{-\frac{11}{4}} \Rightarrow f'''(x_0) = f'''(0) = \frac{21}{64}.
\end{aligned}$$

It follows that:

$$P_3(x) = 1 + \frac{1}{4}\,x - \frac{3}{32}\,x^2 + \frac{7}{128}\,x^3.$$

Moreover, we have:

$$\sqrt[4]{1.5} = f(0.5) \approx P_3(0.5) = \frac{1135}{1024} \approx 1.108398.$$

(a2) (**4 points**).
First of all,

$$R_3(0.5) = f(0.5) - P_3(0.5) = \sqrt[4]{0.5} - \frac{1135}{1024} \approx 0.0017$$

Moreover, according to Taylor's Theorem we have:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}\,x^4$$

where $\xi \in [0, x]$.

We have:

$$f^{(4)}(x) = -\frac{231}{256}\,(1+x)^{-\frac{15}{4}}.$$

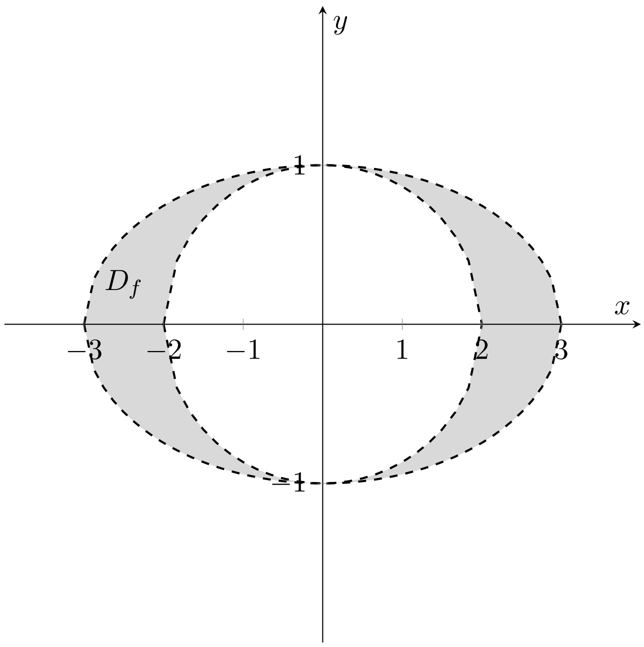
Therefore, for $x \in [0,1]$,

$$
\begin{aligned}
|R_3(x)| &= \frac{\left|f^{(4)}(\xi)\right|}{4!}\, x^4 \\
&= \frac{1}{4!}\,\frac{231}{256}\,(1+\xi)^{\frac{-15}{4}}\, x^4 \\
&= \frac{77}{2048}\quad \underbrace{(1+\xi)^{\frac{-15}{4}}}_{\leq 1 \text{ for } \xi \in [0,1]}\quad \underbrace{x^4}_{\leq 1 \text{ for } x \in [0,1]} \\
&\leq \frac{77}{2048} \\
&\approx 0.037 < 0.04.
\end{aligned}
$$

(b) (**4 points**).
We have:

$$
x \in D_f \Leftrightarrow \left\{ \begin{array}{c} x + 4\,y^2 - 4 > 0 \\ 9 - x^2 - 9\,y^2 > 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} \frac{x^2}{2^2} + \frac{y^2}{1^2} > 1\ (I) \\ \frac{x^2}{3^2} + \frac{y^2}{1^2} < 1\ (II) \end{array} \right. .
$$

This corresponds to the area inside an ellipse (II) with centre $(0,0)$ and semi-axis $a = 3$ and $b = 1$ intersected with the area outside an ellipse (I) with centre $(0,0)$ and semi-axis $a = 2$ and $b = 1$. The following figure illustrates the domain of f:



(c) (**6 points**).
The plane Γ can be written as

$$\Gamma\,:\,z = 10\,x - 2\,y - 14.$$

Thus we define the function g by $g(x,y) = 10\,x - 2\,y - 14$ and

$$\Gamma\,:\,z = g(x,y).$$

The following three conditions must hold:

(i) $P = (2,-1,z_0)$ belongs to Γ and thus $z_0 = g(-2,1) = 8$.

(ii) $P = (2,-1,z_0)$ belongs to the surface of f, and thus

$$z_0 = f(2,-1) = 2^3 + a\cdot 2\cdot(-1) + b\cdot(-1)^2 + c \Leftrightarrow 8 = 8 - 2\,a + b + c \Leftrightarrow c = 2\,a - b.$$

(iii) The functions f and g possess the same partial derivatives at $(x_0,y_0) = (2,-1)$. We have:

$$f_x(x,y) = 3\,x^2 + a\,y$$

and

$$g_x(x,y) = 10.$$

Therefore,

$$f_x(2,-1) = g_x(2,-1) \Leftrightarrow 12 - a = 10 \Leftrightarrow a = 2.$$

Moreover:

$$f_y(x,y) = a\,x + 2\,b\,y$$

and

$$g_y(x,y) = -2.$$

Therefore,

$$f_y(2,-1) = g_y(2,-1) \Leftrightarrow 2\,a - 2\,b = -2 \Leftrightarrow 2\,b = 2\,a + 2 \overset{a=2}{\Leftrightarrow} b = 3.$$

Finally, from (ii) we have $c = 1$.

(d) (**6 points**).
Because f is homogeneous of degree 2, then the Euler equation implies:

$$x\, f_x(x,y) + y\, f_y(x,y) = 2\, f(x,y).$$

Because

$$f_x(x,y) = \frac{2}{x^3}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + 2\,x$$

and

$$f_y(x,y) = \frac{2}{y^3}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + 2\,y$$

we obtain:

$$\begin{aligned}
2\, f(x,y) &= x\, f_x(x,y) + y\, f_y(x,y) \\
&= x\left[\frac{2}{x^3}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + 2\,x\right] + y\left[\frac{2}{y^3}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + 2\,y\right] \\
&= \frac{2}{x^2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + \frac{2}{y^2}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + 2\,x^2 + 2\,y^2 \\
&= 2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-2} + 2\,x^2 + 2\,y^2 \\
&= 2\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-1} + 2\,x^2 + 2\,y^2.
\end{aligned}$$

It follows that:

$$f(x,y) = \left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-1} + x^2 + y^2.$$

Part II: Multiple-choice questions

Exercise 3

	(a)	(b)	(c)	(d)
1.	☐	☐	☒	☐
2.	☐	☒	☐	☐
3.	☐	☐	☐	☒
4.	☐	☐	☒	☐
5.	☐	☒	☐	☐
6.	☐	☒	☐	☐
7.	☒	☐	☐	☐
8.	☐	☒	☐	☐

1. Answer is (c). The implication $B \Rightarrow A \wedge B$ is wrong if and only if B is true and A is false.

2. Answer is (b). We have:

$$\begin{aligned}
\lim_{n\to\infty}\left(1-\frac{1}{2\,n}\right)^{3\,n} &= \lim_{n\to\infty}\left[\left(1-\frac{1}{2\,n}\right)^{n}\right]^3 \\
&= \lim_{n\to\infty}\left[\left(1+\frac{(-0.5)}{n}\right)^{n}\right]^3 \\
&= \left[\lim_{n\to\infty}\left(1+\frac{(-0.5)}{n}\right)^{n}\right]^3 \\
&= \left(e^{-0.5}\right)^3 \\
&= e^{-1.5} \\
&= \frac{1}{\sqrt{e^3}}.
\end{aligned}$$

3. Answer is (d). We have:

$$\lim_{x\to 1}\frac{(x-1)^2}{1+\cos(\pi\,x)} \overset{\text{de l'Hôpital}}{=} \lim_{x\to 1}\frac{2\,(x-1)}{-\pi\,\sin(\pi\,x)} \overset{\text{de l'Hôpital}}{=} \lim_{x\to 1}\frac{2}{-\pi^2\,\cos(\pi\,x)} = \frac{2}{\pi^2}.$$

4. Answer is (c). The present value of project I is $-180,000+\frac{220{,}000}{(1+i)^2}$. The present value of project II is $-200,000+\frac{240{,}000}{(1+i)^2}$. Therefore, project I is preferred to project II when

$$-180,000+\frac{220,000}{(1+i)^2} > -200,000+\frac{240,000}{(1+i)^2} \Leftrightarrow 20,000 > \frac{20,000}{(1+i)^2}.$$

The latter inequality is always satisfied for $i>0$.

5. Answer is (b). We have:

$$\begin{aligned} 2\,\log_a(x) = 3\,\log_a(4) \\ \Leftrightarrow \quad & \log_a(x^2) = \log_a(4^3) \\ \Leftrightarrow \quad & x^2 = 4^3 = 2^6 \\ \overset{x \in \mathbb{R}_{++}}{\Leftrightarrow} \quad & x = 2^3 = 8. \end{aligned}$$

6. Answer is (b). Clearly, f is continuous for $x \neq \frac{\pi}{4}$. Moreover,

$$\lim_{x \to \frac{\pi}{4}} f(x) = \lim_{x \to \frac{\pi}{4}} \frac{\sin(x) - \cos(x)}{a\,\left(x - \frac{\pi}{4}\right)} \overset{\text{de l'Hôpital}}{=} \lim_{x \to \frac{\pi}{4}} \frac{\cos(x) + \sin(x)}{a} = \frac{\sqrt{2}}{a}.$$

It follows that f is continuous in $x = \frac{\pi}{4}$ if and only if

$$\frac{\sqrt{2}}{a} = 1 \Leftrightarrow a = \sqrt{2}.$$

7. Answer is (a). f' is negative if and only if f is monotonically decreasing. Comparing the figures, it is clear that only figure (a) satisfies this property.

8. Answer is (b). Because $\varepsilon_f(t) = t\,\rho_f(t)$, then

$$\varepsilon_f(t) = t\,\left(t\,\ln(t) + e^{4\,t}\right) = t^2\,\ln(t) + t\,e^{4\,t}.$$

This corresponds to answer (b).

Exercise 4

	(a)	(b)	(c)	(d)
1.	☐	☐	☒	☐
2.	☐	☐	☐	☒
3.	☐	☒	☐	☐
4.	☒	☐	☐	☐
5.	☐	☒	☐	☐
6.	☐	☐	☒	☐
7.	☐	☒	☐	☐
8.	☐	☐	☒	☐

1. Answer is (c). Because

$$\varepsilon_f(x) = x\,\frac{f'(x)}{f(x)} = x\,\frac{2\,x\,e^{-x} - x^2\,e^{-x}}{x^2\,e^{-x}} = \frac{(2-x)\,x^2\,e^{-x}}{x^2\,e^{-x}} = 2 - x,$$

then $\varepsilon_f(x) > 1$ (elastic) for all $x < 1$ and $\varepsilon_f(x) < 1$ (inelastic) for all $x > 1$.

2. Answer is (d). We have:

$$f'(x) = e\,x^{e-1} + (x+1)\,e^x$$

and

$$f''(x) = e\,(e-1)\,x^{e-2} + (x+2)\,e^x.$$

Because $f''(x) > 0$ for all $x \in \mathbb{R}_{++}$, then f has no inflection point.

3. Answer is (b). We have:

$$P_4(x) = f(x_0)+f'(x_0)\,(x-x_0)+\frac{1}{2}\,f''(x_0)\,(x-x_0)^2+\frac{1}{3!}\,f'''(x_0)\,(x-x_0)^3+\frac{1}{4!}\,f^{(4)}(x_0)\,(x-x_0)^4.$$

Therefore, the condition

$$a = \frac{f''(0)}{2!}$$

must hold. We have:

$$f'(x) = 2\,x\,e^x + (x^2+1)\,e^x = (x^2+2\,x+1)\,e^x$$

and

$$f''(x) = (2\,x+2)\,e^x + (x^2+2\,x+1)\,e^x = (x^2+4\,x+3)\,e^x.$$

Therefore, $f''(0) = 3$ and

$$a = \frac{f''(0)}{2!} = \frac{3}{2}.$$

4. Answer is (a). We have:

$$(x,y) \in D_f \Leftrightarrow 9 - 4\,x^2 - 4\,y^2 \geq 0 \Leftrightarrow x^2 + y^2 \leq \frac{9}{4} = \left(\frac{3}{2}\right)^2.$$

This corresponds to the area inside a circle with centre $(0,0)$ and radius $\frac{3}{2}$.

5. Answer is (b). Sufficient is, according to the Implicit Function Theorem (Theorem 11.3), the condition $\varphi_y(x_0, y_0) \neq 0$, which is only reported in answer (b).

6. Answer is (c). For $\lambda > 0$ we have:

$$\begin{aligned} f(\lambda\, x, \lambda\, y) &= 8\left(\frac{5}{(\lambda\, x)^2} + \frac{1}{5\,\lambda\, x\,\lambda\, y}\right)^{-0.5} \\ &= 8\left(\frac{5}{\lambda^2\, x^2} + \frac{1}{5\,\lambda^2\, x\, y}\right)^{-0.5} \\ &= 8\left(\frac{1}{\lambda^2}\right)^{-0.5}\left(\frac{5}{x^2} + \frac{1}{5\, x\, y}\right)^{-0.5} \\ &= \lambda\, 8\left(\frac{5}{x^2} + \frac{1}{5\, x\, y}\right)^{-0.5} \\ &= \lambda\, f(x, y). \end{aligned}$$

It follows that f is homogenous of degree 1, or linear homogeneous.

7. Answer is (b). For $\lambda > 0$ we have:

$$\begin{aligned} f(\lambda\, x, \lambda\, y) &= e\left(1 + \frac{4\,\lambda\, y}{\lambda\, x}\right)\sqrt{7\,(\lambda\, x)^2 + (\lambda\, x)\,(\lambda\, y)} \\ &= e\left(1 + \frac{4\, y}{x}\right)\sqrt{\lambda^2\,(7\, x^2 + x\, y)} \\ &= e\left(1 + \frac{4\, y}{x}\right)\lambda\sqrt{7\, x^2 + x\, y} \\ &= \lambda\, e\left(1 + \frac{4\, y}{x}\right)\sqrt{7\, x^2 + x\, y} \\ &= \lambda\, f(x, y). \end{aligned}$$

It follows that f is homogenous of degree 1, or linear homogeneous.

8. Answer is (c). For $\lambda > 0$ we have:

$$\begin{aligned} f(\lambda\, x, \lambda\, y) &= (\lambda\, x)^{a-1}\,(\lambda\, y)^{a+6} + \sqrt{(\lambda\, x)^2 + (\lambda\, y)^2} \\ &= \lambda^{2\,a+5}\, x^{a-1}\, y^{a+6} + \lambda\sqrt{x^2 + y^2} \end{aligned}$$

Therefore, f is homogeneous if and only if $\lambda^{2\,a+5} = \lambda$, i.e., $2\,a + 5 = 1$. It follows that $a = -2$.