Mathematics A
Alternative Date Exam Autumn Semester 2016

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13 July 2017

Part I: Open questions (50 points)

General instructions for open questions:

(i) Your answers must contain all mathematical steps and computations. A correct use of the mathematical notation is expected and will be part of the evaluation.

(ii) Your answer to a sub-exercise must be reported in the foreseen space for solutions. If this space is not enough, please use the corresponding backside or additional separate sheets. When this is the case, you must clearly indicate that your answer is continued on the corresponding backside or on separate sheets. Additionally, your first and last names must be clearly written on each separate sheet.

(iii) Only answers reported in the foreseen space for solutions will be evaluated. Answers reported on the corresponding backside or on separate sheets will be evaluated only if it is clearly indicated that they are continued there.

(iv) The evaluation of a sub-exercise is done according to the points indicated at the top of the page.

(v) Your final answer to a sub-exercise must contain only a single version.

(vi) Temporary computations or sketches must be reported in separate sheets. These sheets must be clearly indicated as drafts and handed in together with the final solutions.

Exercise 1 (26 points)

(a) (6 points)

For which values of a and b ($a, b \in \mathbb{R}_{++}$) does the series

$$\sum_{k=0}^{\infty} \left(\frac{1-b}{1+2a} \right)^k$$

converge?

Represent graphically the solution set (in a (a,b)-system).
Compute the corresponding limit.

Exercise 1

(a) (additional space for your solution)

Exercise 1

(b) (6 points)

A 16-year annuity immediate of CHF 30,000 (i.e., it pays CHF 30,000 for 16 years at the end of each year) should be transformed into a 10-year annuity due (i..e, it pays the amount B for 10 years at the beginning of each year). The interest rate is $i = 5\%$.
Which amount B will be paid each year?

Exercise 1

(b) (additional space for your solution)

Exercise 1

(c) (4 points)

Compute

$$\lim_{t\to 2} \frac{\sqrt{t+2}-\sqrt{2t}}{\sqrt{t+4}-\sqrt{3t}}.$$

Exercise 1

(c) (additional space for your solution)

Exercise 1

(d1) (4 points)

Given is the function

$$f : D_f \to \mathbb{R},\ x \mapsto y = e^{1-\ln(\sqrt{4-x})}.$$

Determine the domain D_f and the range W_f of f.

Exercise 1

(d2) (3 points)

Given is the function

$$f\,:\,D_f \to \mathbb{R},\; x \mapsto y = e^{1-\ln(\sqrt{4-x})}.$$

Is f strictly monotone (proof)?

Exercise 1

(d3) (3 points)

Given is the function

$$f : D_f \to \mathbb{R}, \quad x \mapsto y = e^{1-\ln(\sqrt{4-x})}.$$

Determine the inverse function f^{-1} of f.

Exercise 2 (24 points)

(a1) (4 points)

Given is the function

$$f : [0, \infty] \to \mathbb{R}, \quad x \mapsto y = \sqrt[4]{1+x} = (1+x)^{\frac{1}{4}}.$$

Determine the third order Taylor polynomial $P_3(x)$ of f at the point $x_0 = 0$.

Use $P_3(x)$ to compute an approximation of $\sqrt[4]{1.5}$.

Exercise 2

(a1) (additional space for your solution)

Exercise 2

(a2) (4 points)

Given is the function

$$f\,:\,[0,\infty]\to\mathbb{R},\quad x\mapsto y=\sqrt[4]{1+x}=(1+x)^{\frac{1}{4}}.$$

Let $R_3(x)$ denote the third order remainder term of f at x_0.

Using the definition of the remainder term compute $R_3(0.5)$ with a precision of 4 decimal digits.

Show that for $x\in[0,1]$ we have:

$$|R_3(x)|<0.04.$$

Exercise 2

(b) (4 points)

Given is the function

$$f(x,y) = \ln(x^2 + 4y^2 - 4) + \frac{y-2}{\sqrt{9 - x^2 - 9y^2}}.$$

Determine the domain D_f of f and represent it graphically.

Exercise 2

(b) (additional space for your solution)

Exercise 2

(c) (6 points)

Given is the function

$$f(x, y) = x^3 + axy + by^2 + c.$$

For which values of the parameters a, b, $c \in \mathbb{R}$ is the equation of the tangent plane Γ to the graph of f at the point $P = (2, -1, z_0)$ given by

$$\Gamma : \; 5x - y - \frac{1}{2}z - 7 = 0\,?$$

Exercise 2

(c) (additional space for your solution)

Exercise 2

(d) (6 points)

The function $f(x, y)$ is defined for $x > 0$ and $y > 0$ and is homogenous of degree $k = 2$. The partial derivatives of $f(x, y)$ are

$$f_x(x, y) = \frac{2}{x^3}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-2} + 2x$$

and

$$f_y(x, y) = \frac{2}{y^3}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-2} + 2y.$$

Determine $f(x, y)$ and simplify the term of the function.

Exercise 2

(d) (additional space for your solution)

Part II: Multiple-choice question (50 points)

General instructions for multiple-choice questions:

(i) The solution must be reported in the multiple-choice solution form, which is distributed together with the exam. Only the answers reported in the multiple-choice solution form will be evaluated. The place under the questions is only meant for your notes, but will not be corrected.

(ii) For each question exactly one answer is correct. Therefore, for each question only one possibility can be marked.

(iii) When two or more answers are marked, then the question will be evaluated with 0 points, even if the correct answer is among the marked answers.

(iv) Please carefully read the questions.

Exercise 3 (24 points)

Question 1 (2 points)

The proposition $B \Rightarrow A \wedge B$ has the truth table

(a)

A	T	T	F	F
B	T	F	T	F
$B \Rightarrow A \wedge B$	T	F	F	F

(b)

A	T	T	F	F
B	T	F	T	F
$B \Rightarrow A \wedge B$	T	T	T	T

(c)

A	T	T	F	F
B	T	F	T	F
$B \Rightarrow A \wedge B$	T	T	F	T

(d) None of the above truth tables is correct.

Exercise 3

Question 2 (3 points)

Given is the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by

$$a_n = \left(1 - \frac{1}{2n}\right)^{3n}.$$

(a) $\{a_n\}_{n\in\mathbb{N}}$ has limit 0.

(b) $\{a_n\}_{n\in\mathbb{N}}$ has limit $\frac{1}{\sqrt{e^3}}$.

(c) $\{a_n\}_{n\in\mathbb{N}}$ has limit $e^{1.5}$.

(d) $\{a_n\}_{n\in\mathbb{N}}$ is divergent.

Exercise 3

Question 3 (3 points)

The limit

$$\lim_{x \to 1} \frac{(x-1)^2}{1+\cos(\pi\, x)}$$

is

(a) 2.

(b) $-\frac{\pi}{2}$.

(c) $\frac{\pi}{2}$.

(d) $\frac{2}{\pi^2}$.

Exercise 3

Question 4 (3 points)

An investor can choose between two projects:
Project I requires an investment today of CHF $180,000$ and pays off CHF $220,000$ in two years.
Project II requires an investment of CHF $200,000$ and pays off CHF $240,000$ in 2 years.

For which values of the annual interest rate $i > 0$ should Project I be preferred to Projekt II?

Project I should be preferred to Project II

(a) only for $i < 10\%$.

(b) only for $i > 10\,\%$.

(c) for all $i > 0$.

(d) for no $i > 0$.

Exercise 3

Question 5 (3 points)

The equation

$$2 \log_a(x) = 3 \log_a(4)$$

has (for $x \in \mathbb{R}_{++}$) the following solution set:

(a) $\{6\}$.

(b) $\{8\}$.

(c) $\{\sqrt[3]{16}\}$.

(d) The solution set depends on the base $a \in \mathbb{R}_{++} \setminus \{1\}$.

Exercise 3

Question 6 (4 points)

Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto y = \begin{cases} \frac{\sin(x)-\cos(x)}{a\,(x-\frac{\pi}{4})} & \text{for } x \neq \frac{\pi}{4} \\ 1 & \text{for } x = \frac{\pi}{4} \end{cases}.$$

For which $a \in \mathbb{R}$ is f overall continuous?

(a) $a = -\frac{4}{\pi}$.

(b) $a = \sqrt{2}$.

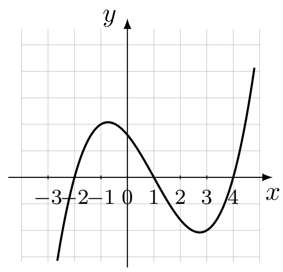
(c) $a = 1$.

(d) f is for no $a \in \mathbb{R}$ overall continuous.

Exercise 3

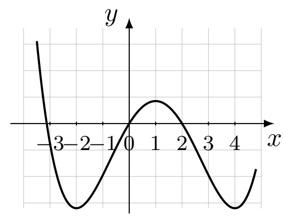
Question 7 (3 points)

Given is the graph of the first derivative f' of function f.



Which of the following figures shows the graph of f?

(a)



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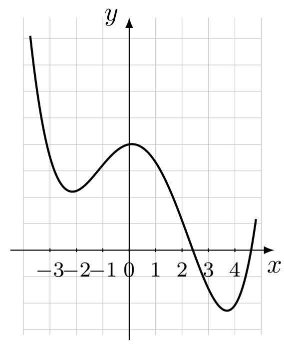
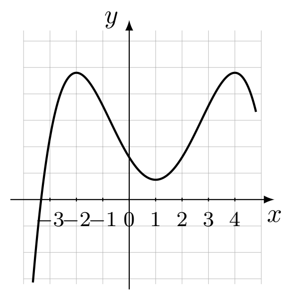
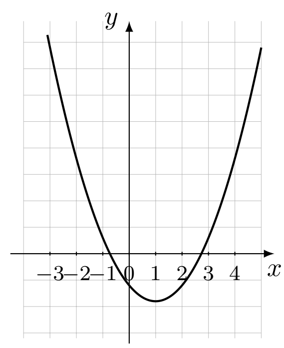


(c)



(d)



Exercise 3

Question 8 (3 points)

For a function f the rate of change $\rho_f(t)$ is:

$$\rho_f(t) = t\,\ln(t) + e^{4t}.$$

The elasticity $\varepsilon_f(t)$ of f satisfies:

(a) $\varepsilon_f(t) = \ln(t) + \frac{e^{4t}}{t}$.

(b) $\varepsilon_f(t) = t^2\,\ln(t) + t\,e^{4t}$.

(c) $\varepsilon_f(t) = \ln(t) + 1 + 4\,e^{4t}$.

(d) None of the above answers is correct.

Exercise 4 (26 points)

Question 1 (3 points)

Given is the function

$$f : \mathbb{R}_{++} \to \mathbb{R}, \quad x \mapsto f(x) = x^2 e^{-x}.$$

(a) f is elastic for $x > 0$.

(b) f is elastic for $x < 2$ and inelastic for $x > 2$.

(c) f is inelastic for $x > 1$ and elastic for $x < 1$.

(d) None of the above answers is correct.

Exercise 4

Question 2 (4 points)

Given is the funtion

$$f : \mathbb{R}_{++} \to \mathbb{R}_{++}, \quad x \mapsto f(x) = x^e + x\,e^x.$$

(a) f has an inflection point at $x_0 = 1$.

(b) f has an inflection point at $x_0 = \sqrt{e}$.

(c) f has an inflection point at $x_0 = e$.

(d) f has no inflection point.

Exercise 4

Question 3 (4 points)

For which $a \in \mathbb{R}$ is

$$P_4(x) = \frac{13}{24}\,x^4 + \frac{7}{6}\,x^3 + a\,x^2 + x + 1$$

the fourth order Taylor polynomial of the function

$$f(x) = (x^2 + 1)e^x$$

at the point $x_0 = 0$?

It is:

(a) $a = 1$.

(b) $a = \frac{3}{2}$.

(c) $a = 2$.

(d) $a = 3$.

Exercise 4

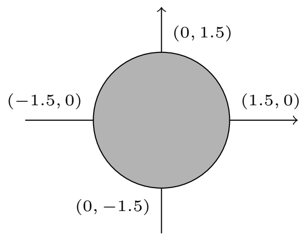
Question 4 (4 points)

Given is the function

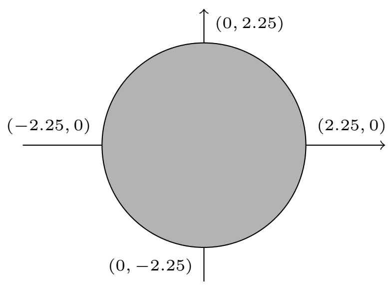
$$f(x, y) = \sqrt{9 - 4x^2 - 4y^2}.$$

Which figure shows the domain of f?

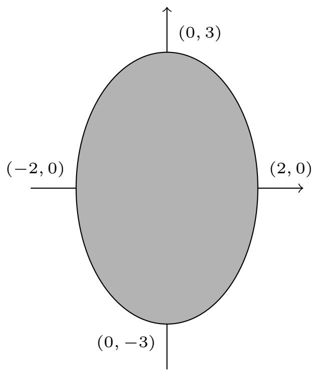
(a)



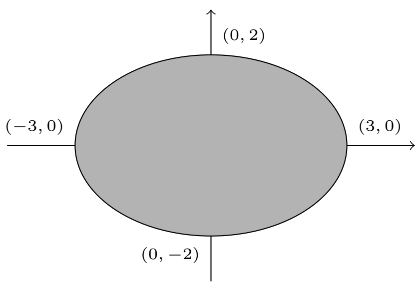
(b)



(c)



(d)



Exercise 4

Question 5 (2 points)

The function $\varphi : D_\varphi \to \mathbb{R}$, $D_\varphi \subset \mathbb{R}^2$, is continuous and $\varphi(x_0, y_0) = 0$. Moreover, φ is continuously differentiable. In order that an open interval $I \subset \mathbb{R}$ with $x_0 \in I$ and a continuously differentiable function $f : I \to \mathbb{R}$ with $\varphi(x, f(x)) = 0$ for all $x \in I$ exist, it is sufficient that

(a) $\varphi_x(x_0, y_0) \neq 0$.

(b) $\varphi_x(x_0, y_0) = 0$ and $\varphi_y(x_0, y_0) \neq 0$.

(c) $\varphi_x(x_0, y_0) = 0$ or $\varphi_y(x_0, y_0) = 0$.

(d) None of the above answers is correct.

Exercise 4

Question 6 (3 points)

Given is the function

$$f(x, y) = 8\left(\frac{5}{x^2} + \frac{1}{5xy}\right)^{-0.5} \quad (x > 0,\ y > 0).$$

(a) f is homogeneous of degree -0.5.

(b) f is homogeneous of degree 0.5.

(c) f is linear homogeneous.

(d) f is not homogeneous.

Exercise 4

Question 7 (3 points)

Given is the function

$$f(x, y) = \frac{e^x}{e^{x-1}} \left(1 + \frac{4y}{x}\right) \left(\sqrt{7\,x^2 + xy}\right) \quad (x > 0,\ y > 0).$$

(a) f is homogeneous of degree 0.

(b) f is linear homogeneous.

(c) f is homogeneous of degree 2.

(d) f is not homogeneous.

Exercise 4

Question 8 (3 points)

For which value of $a \in \mathbb{R}$ is the function

$$f(x, y) = x^{a-1}y^{a+6} + \sqrt{x^2 + y^2} \quad (x > 0,\ y > 0)$$

homogeneous?

(a) f is homogeneous for $a = 0$.

(b) f is homogeneous for $a = 1$.

(c) f is homogeneous for $a = -2$.

(d) f is for no $a \in \mathbb{R}$ homogeneous.